2. *Theorems on Limits of Recurrent Sequences.*

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The object of this paper is to prove some theorems connected with Mercer's theorem[1] by applying Toeplitz's theorem. In §1, we prove a theorem due to Copson and Ferrar,[2] and in §2, a theorem due to Walsh.[3] Although Mr. Walsh applies himself Toeplitz's theorem, his method is much complicated than mine, and is therefore unable to give conditions for the general case

$$y_n=(1+a_n^{(1)})t_n-a_n^{(1)}t_{n-1}-a_n^{(2)}t_{n-2}-\cdots\cdots-a_n^{(m)}t_{n-m},$$

in such a simple form as in §3.

1. *Theorem I. Let*

(i) $a_n>0$ *for all* n,

(ii) $\sum_{n=1}^{\infty}\frac{1}{a_n}$ *diverge.*

Then if $y_n=(1+a_n)t_n-a_nt_{n-1}, \quad y_n=o(1)$,

then $t_n=o(1)$.

Proof. If we put

$$t_0=0, \qquad \alpha_n=\frac{1}{a_n} \qquad (n=1, 2, \ldots\ldots)$$

we have
$$t_n=\frac{\alpha_n}{1+\alpha_n}y_n+\frac{1}{1+\alpha_n}t_{n-1} \qquad (n=1, 2, \ldots\ldots)$$

and then

$$t_n=\frac{\alpha_n}{1+\alpha_n}y_n+\frac{\alpha_{n-1}}{(1+\alpha_n)(1+\alpha_{n-1})}y_{n-1}+\cdots\cdots+\frac{\alpha_1}{(1+\alpha_n)\cdots\cdots(1+\alpha_1)}y_1$$
$$=\frac{\alpha_1}{\prod_{\nu=1}^{n}(1+\alpha_\nu)}y_1+\frac{\alpha_2}{\prod_{\nu=2}^{n}(1+\alpha_\nu)}y_2+\cdots\cdots+\frac{\alpha_n}{1+\alpha_n}y_n.$$

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1) J. Mercer: On the limits of real variants, Proc. London Math. Soc., (2) **5** (1907), pp. 206–224.

2) Copson and Ferrar: Notes on the structure of sequences, Journ. London Math. Soc., **4** (1929), pp. 258–264.

cf. S. Izumi: A theorem on limits and its application, Tohoku Math. Journ., **33** (1931), pp. 181–186.

J. Karamata: Sur quelques inversions d'une proposition de Cauchy et leurs généralisations, ibid., **36** (1932), pp. 22–28.

3) E. Walsh: A note on sequences determined by a recurrence relation, Proc. Edinburgh Math. Soc., (2) **3** (1932), pp. 147–150.

By (ii), we have

$$\prod_{\nu=1}^{n}(1+\alpha_\nu) \to \infty, \qquad \text{as} \quad n \to \infty,$$

and

$$(1) \qquad \frac{\alpha_1}{\prod_{\nu=1}^{n}(1+\alpha_\nu)} + \frac{\alpha_2}{\prod_{\nu=2}^{n}(1+\alpha_\nu)} + \cdots\cdots + \frac{\alpha_n}{1+\alpha_n} < K \qquad \text{(a constant)}.$$

For, the left hand side of (1) is equal to

$$\frac{\prod_{\nu=1}^{n}(1+\alpha_\nu)-1}{\prod_{\nu=1}^{n}(1+\alpha_\nu)} \to 1, \qquad \text{as} \quad n \to \infty.$$

Since the conditions in Toeplitz's theorem are satisfied, we can conclude that $\quad t_n = o(1)$.

2. *Theorem II. Let*

$$\text{(i)} \quad a_n > 0 \qquad \textit{for all } n,$$

$$\text{(ii)} \quad \sum_{n=1}^{\infty} \frac{1}{a_n} \qquad \textit{diverge},$$

$$\text{(iii)} \quad \sum_{n=2}^{\infty} \left| \frac{b_n}{a_n}\left(1+\frac{1}{a_{n-1}}\right) \right| \qquad \textit{converge}.$$

Then if $\quad y_n = (1+a_n)t_n - a_n t_{n-1} - b_n t_{n-2}, \qquad y_n = o(1),$

then $\quad t_n = o(1)$.

Proof. If we put

$$t_0 = t_{-1} = 0$$

and

$$\alpha_n = \frac{1}{a_n}, \qquad \beta_n = b_n \cdot \alpha_n(1+\alpha_{n-1}),$$

then we have $\quad t_n = \dfrac{\alpha_n}{1+\alpha_n} y_n + \dfrac{1}{1+\alpha_n} t_{n-1} + \dfrac{\beta_n}{(1+\alpha_n)(1+\alpha_{n-1})} t_{n-2}.$

Solving this for t_n, we can put

$$t_n = A_n^n y_n + A_n^{n-1} y_{n-1} + \ldots\ldots + A_n^r y_r + \cdots\cdots + A_n^1 y_1.$$

Here

$$(1) \qquad |A_n^r| \leqq \frac{\alpha_r \prod_{\nu=r+2}^{n}(1+|\beta_\nu|)}{\prod_{\nu=r}^{n}(1+\alpha_\nu)},$$

where the product in the numerator is supposed to be equal to 1, for

$$n \geqq r \geqq n-1.$$

For, (1) is obvious, when $n=r, \ r+1$.

Suppose that $m > r$ and (1) be true when $n < m$. Then

$$A_m^r=\frac{1}{1+\alpha_m}A_{m-1}^r+\frac{\beta_m}{(1+\alpha_m)(1+\alpha_{m-1})}A_{m-2}^r$$

$$(m=2,\ 3,\ \ldots\ldots;\quad r=1,\ 2,\ \ldots\ldots,\ m-1),$$

$$|A_m^r|\leqq\frac{\alpha_r\prod_{\nu=r+2}^{m-1}(1+|\beta_\nu|)}{(1+\alpha_m)\prod_{\nu=r}^{m-1}(1+\alpha_\nu)}+\frac{\alpha_r|\beta_m|\prod_{\nu=r+2}^{m-2}(1+|\beta_\nu|)}{(1+\alpha_m)(1+\alpha_{m-1})\prod_{\nu=r}^{m-2}(1+\alpha_\nu)}$$

$$=\frac{\alpha_r\{\prod_{\nu=r+2}^{m-1}(1+|\beta_\nu|)+|\beta_m|\prod_{\nu=r+2}^{m-2}(1+|\beta_\nu|)\}}{\prod_{\nu=r}^{m}(1+\alpha_\nu)}$$

$$\leqq\frac{\alpha_r\prod_{\nu=r+2}^{m}(1+|\beta_\nu|)}{\prod_{\nu=r}^{m}(1+\alpha_\nu)}.$$

Thus, (1) is proved in general.

In virtue of (iii),

$$\prod_{\nu=r+2}^{n}(1+|\beta_\nu|)<M.$$

M being a constant, by (ii), we have

$$A_n^r\rightarrow 0\qquad\text{as } n\rightarrow\infty,\ \text{for a fixed } r,$$

and
$$\sum_{r=1}^{n}|A_n^r|<K\qquad\text{(a constant)}.$$

Thus we can apply Toeplitz's theorem, which leads to

$$t_n=o(1).$$

3. *Theorem III. Let*

(i) $a_n^{(1)}>0$ *for all* n,

(ii) $\sum_{n=1}^{\infty}\frac{1}{a_n^{(1)}}$ *diverge,*

(iii) $\sum_{n=r}^{\infty}\left|\frac{a_n^{(r)}}{a_n^{(1)}}\prod_{s=1}^{r-1}\left(1+\frac{1}{a_{n-s}^{(1)}}\right)\right|$ $(r=2,\ 3,\ \ldots\ldots,\ m)$ *converge.*

Then if $y_n=(1-a_n^{(1)})t_n-a_n^{(1)}t_{n-1}-a_n^{(2)}t_{n-2}-\cdots\cdots-a_n^{(m)}t_{n-m}$

and $y_n=o(1)$,

then $t_n=o(1)$.

Proof. If we put

$$t_0=t_{-1}=t_{-2}=\cdots\cdots=t_{-m+1}=0$$

and
$$\alpha_n^{(1)}=\frac{1}{a_n^{(1)}},\qquad \alpha_n^{(r)}=a_n^{(r)}\cdot\alpha_n^{(1)}(1+\alpha_{n-1}^{(1)})\cdots\cdots(1+\alpha_{n-r+1}^{(1)})$$

$$(r=2,\ 3,\ \ldots\ldots,\ m),$$

then we have

$$t_n=\frac{a_n^{(1)}}{1+a_n^{(1)}}y_n+\frac{1}{1+a_n^{(1)}}t_{n-1}+\frac{a_n^{(2)}}{(1+a_n^{(1)})(1+a_{n-1}^{(1)})}t_{n-2}+\cdots$$
$$\cdots+\frac{a_n^{(m)}}{(1+a_n^{(1)})(1+a_{n-1}^{(1)})\cdots\cdots(1+a_{n-m+1}^{(1)})}t_{n-m}.$$

Eliminating t's, we can put

$$t_n=A_n^n y_n+A_n^{n-1}y_{n-1}+\cdots\cdots+A_n^r y_r+\cdots\cdots+A_n^1 y_1.$$

Here

$$|A_n^r|\leqq\frac{a_r^{(1)}\prod\limits_{\mu=2}^{m}\{\prod\limits_{\nu=r+\mu}^{n}(1+|a_\nu^{(\mu)}|)\}}{\prod\limits_{\nu=r}^{n}(1+a_\nu^{(1)})},\tag{1}$$

where, for $n\geqq r\geqq n-\mu+1$, the product in the numerator including these suffixes is supposed to be equal to 1.

For, (1) is obvious, when $n=r$, $r+1$.

Suppose that $k>r$ and (1) be true when $n<k$. Then

$$A_k^r=\frac{1}{1+a_k^{(1)}}A_{k-1}^r+\frac{a_k^{(2)}}{(1+a_k^{(1)})(1+a_{k-1}^{(1)})}A_{k-2}^r+\cdots$$
$$\cdots+\frac{a_k^{(m)}}{(1+a_k^{(1)})\cdots\cdots(1+a_{k-m+1}^{(1)})}A_{k-m}^r,$$
$$(k=2,3,\ldots\ldots;\ r=1,2,\ldots\ldots,k-1),$$

where A_k^r for negative k is supposed to be 0.

$$|A_k^r|\leqq\frac{a_r^{(1)}}{\prod\limits_{\nu=r}^{k}(1+a_\nu^{(1)})}\Big(\prod_{\mu=2}^{m}\{\prod_{\nu=r+\mu}^{k-1}(1+|a_\nu^{(\mu)}|)\}+|a_k^{(2)}|\prod_{\mu=2}^{m}\{\prod_{\nu=r+\mu}^{k-2}(1+|a_\nu^{(\mu)}|)\}+\cdots$$
$$\cdots+|a_k^{(m)}|\prod_{\mu=2}^{m}\{\prod_{\nu=r+\mu}^{k-m}(1+|a_\nu^{(\mu)}|)\}\Big)$$
$$\leqq\frac{a_r^{(1)}\prod\limits_{\mu=2}^{m}\{\prod\limits_{\nu=r+\mu}^{k}(1+|a_\nu^{(\mu)}|)\}}{\prod\limits_{\nu=r}^{k}(1+a_\nu^{(1)})}.$$

Thus (1) is proved in general.

Therefore $\quad A_n^r\to 0 \quad$ as $n\to\infty$, for a fixed r,

and $\quad \sum\limits_{\nu=r}^{n}|A_n^r|<K \quad$ (a constant).

Thus the theorem is proved.

Remark. If conditions (i) *and* (iii) *in the Theorem III still hold and* $a_n^{(r)}\geqq 0$, *then* (ii) *is necessary for* $t_n=o(1)$, *for any sequence* (y_n), *as defined above.*