PAPERS COMMUNICATED

13. *The Foundation of the Theory of Displacements, II.*

(*Application to the functional manifolds.*)

By Akitsugu KAWAGUCHI.
Mathematical Institute, Hokkaido Imperial University, Sapporo.

(Comm. by M. FUJIWARA, M.I.A., Feb. 12, 1934.)

As applications of the general theory set out in my previous paper[1] I shall take here the functional space manifolds and in the next paper the manifolds of matrices. We may regard the former as of infinitely many dimensions and the latter of indeterminate dimensions.

1. Let the underlying manifold M and the associated manifolds $M^{(i)}$, $\overline{M}$ be all functional manifolds, which consist of summable real functions of a real variable.[2] In this case any element a^* of M^* is a system of functions $a(t)$, $a^{(i)}(t)$, which are elements of M, $M^{(i)}$ respectively. It is natural to take such correspondence between any manifolds $\overline{M}$ as the underlying isomorphism that corresponding functions have same values for all same values of their variables. By the term neighbourhood of a function a^t[3] we understand a totality of functions ξ^t such that $|\xi^t - a^t| < \varepsilon$, ε being a positive number. Consider an element $\bar{a}^t(a^*)$ in $\overline{M}_{a*}$ determined uniquely for every a^* and continuous with respect to a^*, then the covariant change of a function $\bar{a}^t(a^*)$ due to a displacement D_{a*b*} can be represented by

$$\nabla\bar{a}^t = \triangle\bar{a}^t + \Gamma^t(\bar{a}_{a*}, \bar{a}_{b*}, D_{a*b*}), \tag{1}$$

or by notation of differential $\delta\bar{a}^t$ for b^* approaching a^* such that $D_{a*b*} \rightarrow 1$

$$\nabla\bar{a}^t(a^*) = \delta\bar{a}^t(a^*) + \Gamma^t(\bar{a}_{a*}, a^*, D_{a*, a*+\delta a*}). \tag{2}$$

2. Now we take the following four postulates.

The first Postulate: *The displacement is linear.*

$$\nabla(\bar{a}^t(a^*) + \bar{b}^t(a^*)) = \nabla\bar{a}^t(a^*) + \nabla\bar{b}^t(a^*), \tag{3}$$

---

1) A. Kawaguchi: The foundation of the theory of displacements, Proc. **9** (1933), 351–354 (cit. with F.D.I.).

2) For real or complex functions of several real or complex variables we can also establish a similar theory by some modification.

3) We use the notation a^t for a function $a(t)$ of a real variable t as a matter of convenience.

then it follows

$$(4)\qquad \Gamma^t(\bar{a}_{a^*}+\bar{b}_{a^*}, a^*, D_{a^*, a^*+\delta a^*}) = \Gamma^t(\bar{a}_{a^*}, a^*, D_{a^*, a^*+\delta a^*}) + \Gamma^t(\bar{b}_{a^*}, a^*, D_{a^*, a^*+\delta a^*}),$$

i.e. Γ^t must be a linear functional with respect to $\bar{a}^t$.

The second Postulate: *The linear displacement $D_{a^*b^*}$ is defined by*

$$(5)\qquad \bar{a}^t(b^*) = (\Gamma^t(a^*, b^*)+1)\bar{a}^t(a^*) + \Gamma^t_s(a^*, b^*)\bar{a}^s(a^*),$$ [1]

Γ's being summable functions, then

$$(6)\qquad \Gamma^t(\bar{a}_{a^*}, a^*, D_{a^*, a^*+\delta a^*}) = \Gamma^t(a^*, a^*+\delta a^*)\bar{a}^t(a^*) + \Gamma^t_s(a^*, a^*+\delta a^*)\bar{a}^s(a^*),$$

where $\Gamma^t(a^*, a^*+\delta a^*)$ and $\Gamma^t_s(a^*, a^*+\delta a^*)$ tend to zero as $\delta a^* \to 0$, since $D_{a^*b^*} \to 1$ for b^* approaching a^*.

The third Postulate: *Γ's depend upon the first variations of functions a^t and $a^{(i)t}$ only, and are linear with respect to those variations.*

$$(7)\qquad \begin{aligned} &\Gamma(a^*, a^*+\delta a^*) = \Gamma(a^*, \delta a, \delta a^{(i)}), \\ &\Gamma(a^*, \delta_1 a+\delta_2 a, \delta a^{(i)}) = \Gamma(a^*, \delta_1 a, \delta a^{(i)}) + \Gamma(a^*, \delta_2 a, \delta a^{(i)}), \quad \text{etc.} \end{aligned}$$

The fourth Postulate: *Γ's are of the linear differential forms of Fréchet with respect to the variations δa^t and $\delta a^{(i)t}$.*

$$(8)\qquad \begin{aligned} \Gamma^t(a^*, \delta a, \delta a^{(i)}) &= \Gamma^t_{00}(a^*)\delta a^t + \sum_i \overset{i}{\Gamma}{}^t_{00}(a^*)\delta a^{(i)t} \\ &\quad + \Gamma^t_{0s}(a^*)\delta a^s + \sum_i \overset{i}{\Gamma}{}^t_{0s}(a^*)\delta a^{(i)s}, \\ \Gamma^t_s(a^*, \delta a, \delta a^{(i)}) &= \Gamma^t_{s\cdot}(a^*)\delta a^t + \Gamma^t_{s0}(a^*)\delta a^{((s))} + \Gamma^t_{su}(a^*)\delta a^u \\ &\quad + \sum_i (\overset{i}{\Gamma}{}^t_{s\cdot}(a^*)\delta a^{(i)t} + \overset{i}{\Gamma}{}^t_{s0}(a^*)\delta a^{(i)((s))} + \overset{i}{\Gamma}{}^t_{su}(a^*)\delta a^{(i)u}). \end{aligned}$$ [2]

3. We can deduce from this displacement a generalization of Michal,[3] Moisil[4] and the present author,[5] by taking off $a^{(i)}$ and putting

1) The expression $\Gamma^t_s(a^*, b^*)\bar{a}^s(a^*)$ stands for $\int \Gamma^t_s \bar{a}^s ds$ over a fixed interval. We use in general the convention of letting the repetition of an index in a term, some as subscripts and the others as superscripts, stand for a Lebesgue integration over a fixed interval with respect to this index.

2) $\Gamma^t_{s0}\delta a^{((s))}$ means that this term is not integrated with respect to s.

3) A. D. Michal: Function space-time manifolds, Proceedings of the Nat. Academy of Science, **17** (1931), 217-225.

4) G. C. Moisil: Sur les variétés fonctionnelles, Comptes Rendus Paris, **187** (1928), 796-798.

5) A. Kawaguchi: Sur les différentes connexions de l'espace fonctionnel, Comptes Rendus Paris, **189** (1929), 189-191 and Über Übertragungen im Funktionenraume, Comptes Rendus du premier Congrès des Mathématiciens des Pays Slaves, Warszawa 1929, 329-334.

the restriction for a^t as well as $\bar{a}^t$:

$$
(9) \qquad \begin{array}{lll} a^t=\bar{a}^t=0 & \text{for} & t<a,\ t>b+n, \\ a^t=f(t),\ \ \bar{a}^t=\varphi(t) & \text{in a closed interval} & (a, b), \\ a^t=a^\lambda,\ \ \bar{a}^t=\bar{a}^\lambda & \text{for} & b+\lambda-1<t\leqq b+\lambda, \end{array}
$$

a^λ and $\bar{a}^\lambda$ being constants and $\lambda=1, 2, 3, \ldots\ldots, n$. The system of invariants of this displacement is a generalization of that corresponding to Picard's transformation, by which the functionals $\bar{a}^t$ are transformed so that

$$
(10) \qquad '\bar{a}^t=p^t\bar{a}^t+p^t_s\bar{a}^s, \qquad p^t\neq 0,
$$

p's being functionals of a^t. In the latter case the transformation (10) stands for (7) in F.D.I. and the transformation of the parameters Γ are

$$
(11) \qquad '\Gamma^t=p^t\delta' p^t+\Gamma^t, \qquad '\Gamma^t_s=\Phi^t_s p^t+\Phi^u_s p^t_u,
$$

where $\qquad \Phi^t_s=\delta' p^t_s+'p^t_s\Gamma^t+'p^{((s))}\Gamma^t_s+'p^u_s\Gamma^t_u-'p^t_{((s))}(\Gamma^s+p^s\delta' p^s)$

and $\bar{a}^t='p^{t\prime}\bar{a}^t+'p^{t\prime}_s\bar{a}^s$ is the inverse transformation of (10), then the covariant differential of $\bar{a}^t$ is transformed in such a manner as $\bar{a}^t$. The displacement corresponding to Fredholm's transformation can be obtained by putting $p^t\equiv 1$ and a special case of this displacement, for which $\Gamma^t=$const., has discussed by Michal already.[1] When indefinite integrals stand for definite integrals, we have the displacements corresponding to Volterra's transformation. Putting $p^t\equiv 0$, there follows a displacement in a Hilbertian space, from which Vitali's displacement[2] can be deduced by specialization.

4. We consider next another kind of functional manifold. Let M be an ordinary manifold of finite dimensions and $x^\lambda(\lambda=1, 2, \ldots\ldots, n)$ be a coordinate system in it. We take as $\overline{M}$ such a manifold that its element is a system of N functions of m real variables, for example $v^\Lambda(y^1, y^2, \ldots\ldots, y^m)$ $(\Lambda=1, 2, \ldots\ldots, N)$, N and m being finite fixed integers. Under the assumption that the transformation in § 7 in F.D.I. which we name the *fundamental transformation* in $\overline{M}$, is linear homogeneous

$$
(12) \qquad 'v^\Lambda=P^\Lambda_M v^M,
$$

where P^Λ_M are functions of x's and y's and its determinant does not

1) A. D. Michal: Affinely connected function space manifold, American Journal of Math., **50** (1928), 473–517.

2) G. Vitali: Geometria nello spazio hilbertiano, Atti d. R. Istituto Veneto, **87** (1927–28), 349–428 and Sopra una derivazione covariante nel calcolo assoluto generalizzato, Rendiconti della R. Accademia Naz. dei Lincei, serie 6, **6** (1927), 202–206 and 278–282, etc.

vanish, the parameters of a linear displacement $\nabla v^{\Lambda}=dv^{\Lambda}+\Gamma^{\Lambda}_{\mathrm{M}}v^{\mathrm{M}}$ are transformed as follows:

$$'\Gamma^{\Lambda}_{\mathrm{M}}=P^{\Lambda}_{\mathrm{N}}(Q^{\Pi}_{\mathrm{M}}\Gamma^{\mathrm{N}}_{\Pi}+dQ^{\mathrm{N}}_{\mathrm{M}}), \tag{13}$$

where $P^{\Lambda}_{\mathrm{N}}Q^{\mathrm{N}}_{\mathrm{M}}=\delta^{\Lambda}_{\mathrm{M}}$. When the fundamental transformation is one due to a change of variables $'y^i='y^i(y^1, y^2, \ldots\ldots, y^m)$ as well as a transformation of the coordinate system $'x^\lambda='x^\lambda(x^1, x^2, \ldots\ldots, x^n\,;\, y^1, y^2, \ldots\ldots, y^m)$ depending on the variables y^i, we shall put $N=n+m$ and $P^{\Lambda}_{\mathrm{M}}=\partial' z^{\Lambda}/\partial z^{\mathrm{M}}$, introducing the notation $z^{\Lambda}=y^i$, for $\Lambda=i$, $=x^\lambda$ for $\Lambda=m+\lambda=\alpha$, then $P^i_\alpha=Q^i_\alpha=0$. The transformation (13) then proceeds to

$$
\begin{aligned}
&'\Gamma^i_j=P^i_k(Q^l_j\Gamma^k_l+Q^\gamma_j\Gamma^k_\gamma+dQ^k_j), \\
&'\Gamma^i_\beta=P^i_kQ^\gamma_\beta\Gamma^k_\gamma, && i,j,k,l=1,2,\ldots\ldots,m \\
&'\Gamma^\alpha_j=P^\alpha_{\mathrm{N}}(Q^{\Pi}_j\Gamma^{\mathrm{N}}_{\Pi}+dQ^{\mathrm{N}}_j), && \alpha,\beta,\gamma,\delta=m+1,\ldots\ldots,m+n \\
&'\Gamma^\alpha_\beta=P^\alpha_iQ^\gamma_\beta\Gamma^i_\gamma+P^\alpha_\delta(Q^\gamma_\beta\Gamma^\delta_\gamma+dQ^\delta_\beta)
\end{aligned}
\tag{14}
$$

and we know, Γ^i_β has a contravariant property with respect to i referred to the change of variables and covariant property with respect to β referred to the transformation of the coordinate system $'x^\lambda='x^\lambda(x^1, x^2, \ldots\ldots, x^n)$. The connections studied by Wundheiler[1] and Hlavatý[2] are both special cases of this displacement, as we can see easily. The former correspond to a moving coordinate system along a curve, i.e. to coordinate systems associated with all points on a curve, and the latter to those associated with all points on a manifold of m dimensions.

5. The last displacement can also be obtained only by consideration of manifolds of finite dimensions, and not of functional manifolds, as follows. Let M be a manifold with a coordinate system y^i, $M^{(1)}$ that with x^λ and an element of $\overline{M}_{\alpha *}$ expressed by N numbers v^{Λ}, then v^{Λ} depends upon x^λ and y^i, where the transformation of the coordinate system x^λ is not independent of y^i in general. From this standpoint the displacements in Finsler's space are also contained in the abovementioned as special cases. This consideration can be extended directly to the case where the transformation of the coordinate system is $'z^{\Lambda}='z^{\Lambda}(z^1, z^2, \ldots\ldots, z^{n_k})$ for $n_{k-1}<\Lambda\leqq n_k$.

---

1) A. Wundheiler: Kovariante Ableitung und die Cesàroschen Unbeweglichkeitsbedingungen, Math. Zeitschrift, **36** (1932), 104–109.

2) V. Hlavatý: Über eine Art der Punktkonnexion, ibid., **38** (1933), 135–145.