85. Remarks to Some Theorems of Burnside.

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Burnside proved the following two theorems concerning the theory of representation of finite order in his Theory of Groups of Finite Order.[1]

Theorem 1. If Γ be a representation of G as a group of linear substitutions, and if G is simply isomorphic with Γ, and when the process of compounding Γ with itself is carried far enough, then, every irreducible representation of G will arise.

Theorem 2. If $s(<r)$ of the irreducible representations of G, viz. Γ_1, Γ_2, ..., Γ_s combine among themselves by composition, then G has a self-conjugate subgroup H, each of whose operation is represented by the identical substitution in these s representations of G and in no others.

To prove these theorems Burnside used the convergence of a power series. In his case the representations of the group are done in the domain of complex numbers. If we take an abstract domain in place of the domain of complex numbers, we must modify slightly his proof. The object of this paper is to give purely algebraic proof to these theorems. In the course of these proofs it is convinced that we can express Theorem 1 more precisely, and further we will see that these two theorems hold good also, even when the group is represented by collineation groups. In the following we will adopt the notations in Prof. Burnside's work.

Proof. If we denote the reduced form of the Kronecker's product Γ^n by the formula

$$\Gamma^n = \sum_i \gamma_{ni} \Gamma_i \,,$$

then

$$(\psi_p)^n = \sum_p h_p \chi_p^i (\psi'_p)^n \,,$$

from which we obtain

$$\text{(A)} \qquad N\gamma_{ni} = \sum_i h_p \chi_p^i (\psi'_p)^n$$

by using the formula

1) p. 298.

(B) $$\sum_{p=1}^{r} h_p \chi_p^{i} \chi_p^{k} = \begin{cases} N & k=i' \\ 0 & k \neq i' . \end{cases}$$

By our assumption the characteristic ψ_1 of E is different from the characteristics $\psi_2, \psi_3, \ldots\ldots, \psi_r$ of other operations of G. Now we will classify these r characteristics in the following r' classes, in which any two characteristics belonging to different classes are not equal and further put

$$\begin{aligned}
&\psi_1 \equiv \psi'_1 , \\
&\psi_2 = \psi_3 = \cdots\cdots = \psi_{l_2} \equiv \psi'_2 , \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\psi_{l_{r'-1}+1} = \psi_{l_{r'-1}+2} = \cdots\cdots = \psi_{l_{r'}} \equiv \psi'_{r'} , \\
&\psi_{l_{r'}+1} = \psi_{l_{r'}+2} = \cdots\cdots = \psi_r \equiv 0 .
\end{aligned}$$

Then $\psi'_1, \psi'_2, \ldots\ldots, \psi'_{r'}$ are all different from zero and any two of them are unequal.

If any irreducible representation Γ_i does not occur in Γ^n for some value of n, or in other words $\gamma_{ni}=0$ for all values of n, we would have the following r' relations from (A):

$$\begin{aligned}
&h_1\chi_1^{i'}\psi'_1 + (h_2\chi_2^{i'} + \cdots + h_{l_2}\chi_{l_2}^{i'})\psi'_2 + \cdots + (h_{l_{r'-1}+1}\chi_{l_{r'-1}}^{i'} + \cdots + h_{l_{r'}}\chi_{l_{r'}}^{i'})\psi'_{r'} = 0 , \\
&h_1\chi_1^{i'}\psi'^2_1 + (h_2\chi_2^{i'} + \cdots + h_{l_2}\chi_{l_2}^{i'})\psi'^2_2 + \cdots + (h_{l_{r'-1}+1}\chi_{l_{r'-1}}^{i'} + \cdots + h_{l_{r'}}\chi_{l_{r'}}^{i'})\psi'^2_{r'} = 0 , \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&h_1\chi_1^{i'}\psi'^{r'}_1 + (h_2\chi_2^{i'} + \cdots + h_{l_2}\chi_{l_2}^{i'})\psi'^{r'}_2 + \cdots + (h_{l_{r'-1}+1}\chi_{l_{r'-1}}^{i'} + \cdots + h_{l_{r'}}\chi_{l_{r'}}^{i'})\psi'^{r'}_{r'} = 0 .
\end{aligned}$$

As $h_1\chi_1^{i'}$ is different from zero, it would follow that the relation

$$\begin{vmatrix}
\psi'_1 & \psi'_2 & \cdots\cdots & \psi'_{r'} \\
\psi'^2_1 & \psi'^2_2 & \cdots\cdots & \psi'^2_{r'} \\
\cdots & \cdots & \cdots & \cdots \\
\psi'^{r'}_1 & \psi'^{r'}_2 & \cdots\cdots & \psi'^{r'}_{r'}
\end{vmatrix}
\begin{aligned}
&= \psi'_1\psi'_2 \cdots\cdots \psi'_{r'}(\psi'_1-\psi'_2)(\psi'_1-\psi'_3) \cdots\cdots (\psi'_1-\psi'_{r'}) \\
&\qquad\qquad \times(\psi'_2-\psi'_3) \cdots\cdots (\psi'_2-\psi'_{r'}) \\
&\qquad\qquad\qquad \cdots\cdots\cdots\cdots \\
&\qquad\qquad\qquad\qquad \times(\psi'_{r'-1}-\psi'_{r'}) \\
&= 0 .
\end{aligned}$$

But this obviously contradicts the definitions of $\psi'_1, \psi'_2, \ldots\ldots, \psi'_{r'}$.

Thus Theorem 1 is proved. In this process of proving we see that all of numbers $\gamma_{1i}, \gamma_{2i}, \ldots\ldots, \gamma_{r'i}$ cannot be equal to zero, so that we can modify Theorem 1 in the following form.

Theorem 1′. Let Γ be any representation of G, which is simply isomorphic with G. If there exist r' unequal characteristics of G in Γ which are not zero, every irreducible representation of G will arise in some of r' Kronecker's products $\Gamma^1, \Gamma^2, \ldots\ldots, \Gamma^{r'}$ as an irreducible component.

For example r' is equal to 1 in the regular representation of G. In fact every irreducible representation occurs in it.

Proof of Theorem 2. The representation $\Gamma = \sum_1^s a_i \Gamma_i$ is not simply isomorphic with G by Theorem 1. Hence there is a maximum self-conjugate subgroup H of G such that to the operations of H there correspond the identical substitutions in each of the representations $\Gamma_1, \Gamma_2, \ldots\ldots, \Gamma_s$.

Suppose now, if possible, that Γ_t $(t > s)$ is another representation of G to the identical substitution of which the operations of H correspond. Since Γ_t does not occur in Kronecker's products Γ^n for any values of n

$$\text{(C)} \qquad N\gamma_{nt} = \sum_p h_p \chi_p^{t'} (\psi_p)^n = 0\,, \qquad n = 1, 2, \ldots\ldots .$$

Now for each conjugate set in H, and for these only

$$\psi_p = \psi_1\,;$$

and for each conjugate set in H

$$\chi_p^{t'} = \chi_1^{t'}\,.$$

As in Theorem 1 we classify r characteristics $\psi_1, \psi_2, \ldots\ldots, \psi_r$ of G in Γ in the r' following classes

$$\begin{aligned}
&\psi_1 = \psi_2 = \ldots\ldots = \psi_{l_1} \equiv \psi_1'\,,\\
&\psi_{l_1+1} = \psi_{l_1+2} = \cdots\cdots = \psi_{l_2} \equiv \psi_2'\,,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\psi_{l_{r'-1}+1} = \psi_{l_{r'-1}+2} = \cdots\cdots = \psi_{l_{r'}} \equiv \psi_{r'}'\,,\\
&\psi_{l_{r'}+1} = \psi_{l_{r'}+2} = \cdots\cdots = \psi_r \equiv 0\,.
\end{aligned}$$

Then from (C) we obtain the following r' relations

$$\begin{aligned}
&n_H \chi_1^{t'} \psi_1' + (h_{l_1+1} \chi_{l_1+1}^{t'} + \cdots + h_{l_2} \chi_{l_2}^{t'}) \psi_2' + \cdots + (h_{l_{r'-1}+1} \chi_{l_{r'-1}+1}^{t'} \\
&\qquad\qquad + \cdots + h_{l_{r'}} \chi_{r'}^{t'}) \psi_{r'}' = 0\,,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&n_H \chi_1^{t'} \psi_1'^{r'} + (h_{l_1+1} \chi_{l_1+1}^{t'} + \cdots + h_{l_2} \chi_{l_2}^{t'}) \psi_2'^{r'} + \cdots + (h_{l_{r'-1}+1} \chi_{l_{r'-1}+1}^{t'} \\
&\qquad\qquad + \cdots + h_{l_{r'}} \chi_{r'}^{t'}) \psi_{r'}'^{r'} = 0\,,
\end{aligned}$$

where n_H denotes the order of H. As in Theorem 1 this is a contradiction, since $n_H\chi_1^{t'}\psi'$ is not zero. Thus Theorem 2 is proved.

In order to extend these two theorems to the case of the representations by collineation groups, we must only use the formula[1]

$$\sum_{\iota_{A_i}} h_{A_i}\chi_\nu(A_i)\chi_\mu(A_i^{-1})=\begin{cases} h & \nu=\mu \\ 0 & \nu\neq\mu \end{cases}$$

instead of the formula B.

1) The Science Reports of the Tohoku Imperial University, 23 (1933), p. 88.