PAPERS COMMUNICATED

83. *On the Convergence Factor of Fourier-Lebesgue Series.*

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1. Let $f(t)$ be a summable periodic function with period 2π, and let

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) ,$$

$$\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\} ,$$

and

$$\phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \phi(u) du .$$

Then we have

Theorem A.[1] *If* $\alpha > 0$ *and*

$$\phi_\alpha(t) = o(t^\alpha) , \tag{1.1}$$

then the series

$$\sum_{n=1}^{\infty} \frac{a_n \cos nt + b_n \sin nt}{n^{\frac{\alpha}{\alpha+1}}}$$

is convergent for $t = x$.

A summable function $f(t)$ is said to belong to L_p, or simply, $f(t) \in L_p$ provided that its p-th power $|f(t)|^p$ is summable in $(-\pi, \pi)$. The object of this paper is to prove some related theorems as Theorem A.

Theorem 1. *If* $f(t) \in L_p$ $(p > 1)$, *and*

$$\int_0^t \phi(t) dt = o(t) , \tag{1.2}$$

then the series

$$\sum_{n=1}^{\infty} \frac{a_n \cos nt + b_n \sin nt}{n^{\frac{1}{p+1}}}$$

is convergent for $t = x$.

Lemma 1. *If the conditions of Theorem 1 are satisfied, then*

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) = o(n^{\frac{1}{p+1}}) .$$

1) F. T. Wang: Tohoku Math. Journ. (under the press).

Proof.

$$\int_0^\eta \phi(t)\frac{\sin nt}{t}dt=\int_0^{r^\delta n^{-\delta}}+\int_{r^\delta n^{-\delta}}^\eta=J_1+J_2, \quad \text{say.} \qquad (1.31)$$

We have

$$J_1=\left[\phi_1(t)\frac{\sin nt}{t}\right]_0^{r^\delta n^{-\delta}}-\int_0^{r^\delta n^{-\delta}}\phi_1(t)\frac{d}{dt}\frac{\sin nt}{t}dt$$

$$=o(1)+o(n\int_0^{r^\delta n^{-\delta}}dt)=o(n^{1-\delta}). \qquad (1.32)$$

We put $\frac{1}{p}+\frac{1}{q}=1$ and use the Hölder's inequality, then we have

$$|J_2|\leqq\int_{r^\delta n^{-\delta}}^\eta \frac{|\phi(t)|}{t}dt$$

$$\leqq\Big(\int_{r^\delta n^{-\delta}}^\eta \frac{1}{t^q}dt\Big)^{\frac{1}{q}}\Big(\int_{r^\delta n^{-\delta}}^\eta |\phi(t)|^p dt\Big)^{\frac{1}{p}}$$

$$\leqq\frac{1}{q-1}\Big(\frac{n}{r}\Big)^{\delta\frac{q-1}{q}}\Big(\int_0^\eta |\phi(t)|^p dt\Big)^{\frac{1}{p}}$$

$$\leqq\frac{1}{q-1}\Big(\frac{n}{r}\Big)^{\frac{\delta}{p}}\Big(\int_0^\eta |\phi(t)|^p dt\Big)^{\frac{1}{p}}.$$

If we take $\delta=\frac{p}{p+1}$, then $1-\delta=\frac{1}{p+1}$, hence

$$s_n(x)=o(n^{\frac{1}{p+1}})+O\Big[\Big(\frac{n}{r}\Big)^{\frac{1}{p+1}}\Big]=o(n^{\frac{1}{p+1}}),$$

when r tends to infinity.

Lemma 2.[1] *If* (1.2) *holds, then*

$$S_n(x)=s_0(x)+s_1(x)+\cdots\cdots+s_n(x)=o(n\log n).$$

Proof of Theorem 1. From Abel's lemma, we have

$$\frac{a_k\cos kx+b_k\sin kx}{k^{\frac{1}{p+1}}}=\sum_{k=n}^{m-2}S_k(x)\Delta^2\frac{1}{k^{\frac{1}{p+1}}}-S_{n-1}(x)\Delta\frac{1}{(n-1)^{\frac{1}{p+1}}}$$

$$+S_{m-1}(x)\Delta\frac{1}{(m-1)^{\frac{1}{p+1}}}-s_{n-1}(x)\frac{1}{(n-1)^{\frac{1}{p+1}}}+s_m(x)\frac{1}{m^{\frac{1}{p+1}}}$$

$$=\sum_{k=n}^{m-2}o(k\log k)o(k^{-2-\frac{1}{p+1}})+o(1)=o(1).$$

Thus the theorem is proved.

1) M. Jacob: Proc. London Math. Soc. (2), vol. 26.

We can construct an even function $\phi(t)$ such that $\phi(t) \in L_p$ and (1.2) holds, but the series

$$\sum_{n=1}^{\infty} \frac{a_n \cos nt + b_n \sin nt}{n^{\delta}}$$

is divergent at $t=x$, for $0 < \delta < \dfrac{1}{p+1}$.

We have defined a function $\phi(t)$, in my paper "On the convergence factor of Fourier series at a point, I" which depends on δ.

Now, if $\delta < \dfrac{1}{p+1}$, then the series

$$\sum_{k=2}^{\infty} \int_{a_k}^{a_{k-1}} |\phi(t)|^p dt \leqq \sum_{k=2}^{\infty} c_k^p \frac{2\pi(2k+1)^{2-2\delta}}{p_k^{2-2\delta}}$$
$$\leqq 2 \sum_{k=2}^{\infty} \frac{(2n+1)^{2-2\delta}}{p_k^{2-2(p+1)\delta}}$$

is convergent. And then $\phi(t) \in L_p$. The other part is the same as there.

2. Theorem 2.[1] *If $f(t) \in L_p$ $(p \geqq 1)$, then the series*

$$\sum_{n=1}^{\infty} \frac{a_n \cos nt + b_n \sin nt}{n^{\frac{1}{p}+\delta}}$$

is convergent for all t and $\delta > 0$.

Proof. Now

$$\int_0^{\eta} \phi(t) \frac{\sin nt}{t} dt \leqq \left(\int_0^{\eta} |\phi(t)|^p dt\right)^{\frac{1}{p}} \left(\int_0^{\eta} \left|\frac{\sin nt}{t}\right|^q dt\right)^{\frac{1}{q}}, \tag{2.1}$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

$$\int_0^{\eta} \left|\frac{\sin nt}{t}\right|^q dt = n^q \int_0^{\frac{1}{n}} o(1) dt + \int_{\frac{1}{n}}^{\eta} o(t^{-q}) dt = o(n^{q-1}). \tag{2.2}$$

From (2.1) and (2.2), we have

$$s_n(x) = o(1) + \frac{1}{\pi} \int_0^{\eta} \phi(t) \frac{\sin nt}{t} dt$$
$$= o(1) + o(n^{\frac{q-1}{q}}) = o(n^{\frac{1}{p}}). \tag{2.3}$$

Therefore

1) The convergence factor in Theorem 2 is certainly replaced by $n^{\frac{1}{p}} (\log n)^{\frac{1}{p}+\delta}$ or more general L function.

$$\sum_{k=n}^{m} \frac{a_k \cos kx + b_k \sin kx}{k^{\frac{1}{p}+\delta}}$$

$$= \sum_{k=n}^{m-1} s_k(x) \Delta k^{-\frac{1}{p}-\delta} + \frac{s_m(x)}{m^{\frac{1}{p}+\delta}} - \frac{s_{n-1}(x)}{(n-1)^{\frac{1}{p}+\delta}}$$

$$= \sum_{k=n}^{m-1} o(k^{\frac{1}{p}}) o(k^{-\frac{1}{p}-1-\delta}) + o(1) = o(1)\,.$$

Thus the theorem is proved.

The positive number δ can not be omitted in the theorem. For, the series

$$\frac{\cos nx}{n^{\frac{1}{q}}(\log n)}, \qquad (1 < q \leqq 2)$$

is, by the Hausdorff's theorem, a Fourier series of function belonging to L_p $(p \geqq 2)$. But the series

$$\frac{\cos nx}{n^{\frac{1}{p}}(n^{\frac{1}{q}} \log n)}$$

is divergent for $x=0$.