28. *On Hansen's Coefficients in the Expansions for Elliptic Motion.*

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Let r be the radius vector, a the semi-major axis, v the true anomaly, ζ the mean anomaly, u the eccentric anomaly, e the eccentricity, and m a positive integer, n an integer, positive or negative. Further put $z=E^{i\zeta}$, where E is the base of Napier's logarithm and $i=\sqrt{-1}$. The coefficients X_j^{nm} in the Laurent expansion of a function:

$$\left(\frac{r}{a}\right)^n E^{imv}=\left(\frac{r}{a}\right)^n(\cos mv+i\sin mv)=\sum_{j=-\infty}^{\infty} X_j^{nm}z^j ,$$

are called Hansen's coefficients and were studied by Tisserand[1] with an elementary but complicated analysis. I propose to deduce the same result by a simpler mode of procedure.

The coefficients can be written

$$X_j^{nm}=\frac{1}{2\pi i}\int_C^{(0+)}\left(\frac{r}{a}\right)^n t^m z^{-j-1}dz ,$$

where $t=E^{iv}$, by the famous Cauchy's theorem of residues in the theory of analytic functions, the contour of integration being taken so as to make a positive circuit round $z=0$ in the ring-domain excepting $z=0$ and $z=\infty$. Now write $s=E^{iu}$ and

$$\omega=\frac{e}{1+\sqrt{1-e^2}}=\frac{1-\sqrt{1-e^2}}{e}<1 ,$$

then Kepler's equation can be transformed into

$$z=sE^{-\frac{e}{2}\left(s-\frac{1}{s}\right)} .$$

By the well-known formula for elliptic motion, we have

$$\frac{r}{a}=1-\frac{e}{2}\left(s+\frac{1}{s}\right)=\frac{1}{1+\omega^2}\left(1-\omega s\right)\left(1-\frac{\omega}{s}\right).$$

Hence

$$X_j^{nm}=\frac{1}{2\pi i}\int_{C_s}^{(0+)}\frac{s^m}{(1+\omega^2)^{n+1}}\left(1-\omega s\right)^{n-m+1}\left(1-\frac{\omega}{s}\right)^{n+m+1}$$
$$\times E^{\frac{j\omega}{1+\omega^2}\left(s-\frac{1}{s}\right)}\cdot s^{-j-1}ds ,$$

1) F. Tisserand: Traité de Mécanique Céleste. T. 1 (1889), Chap. XV.

where the contour C_s is the transformed contour of C from the z-plane to the s-plane. The points $z=\pm 1$ are invariant in the transformation. There is no singularity besides the essential singularities at $s=0$ and $s=\infty$, except $s=\omega$ and $s=\frac{1}{\omega}$, of which either or both may be singular points.

After Hill[2] we introduce the Bessel function so that

$$E^{\frac{j\omega}{1+\omega^2}\left(s-\frac{1}{s}\right)}=\sum_{p=-\infty}^{\infty} J_p\left(\frac{2j\omega}{1+\omega^2}\right)\cdot s^p .$$

Put $s=\omega\sigma$ and denote the transformed contour of C_s by C_σ, then

$$X_j^{nm}=\sum_{p=-\infty}^{\infty} J_p\left(\frac{2j\omega}{1+\omega^2}\right)\cdot X_{jp}^{nm} ,$$

$$X_{jp}^{nm}=\frac{(-1)^{n+m+1}}{(1+\omega^2)^{n+1}}\cdot\frac{\omega^{m+p-j}}{2\pi i}\int_{C_\sigma}^{(0+)}\sigma^{p-n-j-2}(1-\sigma)^{n+m+1}(1-\omega^2\sigma)^{n-m+1}d\sigma .$$

We deform the contour so that it makes a positive circuit round $\sigma=0$ and proceeds to the right along but beneath the real axis to $\sigma=+1$ and there describes a positive circuit round $\sigma=+1$ and returns towards $\sigma=0$ along but above the real axis to the starting point.

Suppose that $m+n+1>-1$ and also for the present that $p-n-j-2$ is complex and its real part is greater than -1. Then $\sigma=0$ is a branch point and $\sigma=+1$ is an ordinary point in the integral for X_{jp}^{nm}. Hence we have along the contour C_σ

$$X_{jp}^{nm}=\frac{(-1)^{n+m+1}}{(1+\omega^2)^{n+1}}\cdot\frac{\omega^{m+p-j}}{2\pi i}\{E^{2\pi i(p-n-j-2)}-1\}\cdot\int_0^1\sigma^{p-n-j-2}(1-\sigma)^{n+m+1}(1-\omega^2\sigma)^{n-m+1}d\sigma .$$

By Euler's representation[3] of a hypergeometric function of Gauss, this is transformed into

$$\begin{aligned}X_{jp}^{nm}&=\frac{(-1)^{n+m+1}}{(1+\omega^2)^{n+1}}\cdot\frac{\omega^{m+p-j}}{2\pi i}\{E^{2\pi i(p-n-j-2)}-1\}\\&\quad\times\frac{\Gamma(p-n-j-1)\Gamma(n+m+2)}{\Gamma(p+m-j+1)}F(m-n-1,\,p-n-j-1,\,p+m-j+1,\,\omega^2),\\&=\frac{(-1)^{n+m+1}}{(1+\omega^2)^{n+1}}\omega^{m+p-j}\cdot E^{\pi i(p-n-j-2)}\frac{\sin\pi(p-n-j-2)}{\pi}\\&\quad\times\frac{\Gamma(p-n-j-1)\Gamma(n+m+2)}{\Gamma(p+m-j+1)}F(m-n-1,\,p-n-j-1,\,p+m-j+1,\,\omega^2).\end{aligned}$$

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2) G. W. Hill: Collected Papers. Vol. 1, p. 221; or H. C. Plummer: An Introductory Treatise on Dynamical Astronomy. (1918), p. 45.

If we use the formula $\frac{\sin \pi\xi}{\pi} = \frac{1}{\Gamma(\xi)\Gamma(1-\xi)}$, we have

$$X_{jp}^{nm} = (-1)^{p+m-j} \frac{\omega^{m+p-j}}{(1+\omega^2)^{n+1}} \cdot \frac{\Gamma(n+m+2)}{\Gamma(-p+n+j+2)\Gamma(p+m-j+1)}$$
$$\cdot F(m-n-1,\ p-n-j-1,\ p+m-j+1,\ \omega^2).$$

This result is true by the theory of analytic continuation for the values of $p-n-j-2$ with its real part less than or equal to -1. Hence we have this expression for X_{jp}^{nm} for all values of $p-n-j-2$ such that $p-n-j-2<0,\ n+m+1>0$.

If $p-n-j-2>0,\ n+m+1>0$, then there is no singularity inside the above contour of integration and we have simply

$$X_{jp}^{nm} = 0.$$

If $p-n-j-2>0,\ n+m+1<0$, then a singularity occurs at $\sigma=1$. In a similar way to the above we get

$$X_{jp}^{nm} = \frac{\omega^{m+p-j}}{(1+\omega^2)^{n+1}} \cdot \frac{\Gamma(p-n-j-1)}{\Gamma(p+m-j+1)\Gamma(-n-m-1)}$$
$$\cdot F(m-n-1,\ p-n-j-1,\ p+m-j+1,\ \omega^2).$$

However $\gamma-\alpha-\beta$ in the expression $F(\alpha\beta\gamma x)$ ought not to be negative. Hence by a famous formula[3] of the hypergeometric functions:

$$F(\alpha\beta\gamma x) = (1-x)^{\gamma-\alpha-\beta}\ F(\gamma-\beta,\ \gamma-\alpha,\ \gamma,\ x),$$

we transform it into

$$X_{jp}^{nm} = \frac{\omega^{m+p-j}(1-\omega^2)^{2n+3}}{(1+\omega^2)^{n+1}} \cdot \frac{\Gamma(p-n-j-1)}{\Gamma(p+m-j+1)\Gamma(-n-m-1)}$$
$$\times F(p+n-j+2,\ -m-n-2,\ p+m-j+1,\ \omega^2).$$

When the exponents are all integers, then the hypergeometric functions contain only finite number of terms.

A similar treatment can be applied in the t-plane.

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3) L. Schlesinger: Einführung in die Theorie der gewöhnlichen Differentialgleichungen auf funktionentheoretischer Grundlage. (1922) s. 234.

F. Klein: Vorlesungen über die hypergeometrischen Funktionen. (1933) s. 62 et suiv.

E. T. Whittaker and G. N. Watson: A Course of Modern Analysis. (1927) Chap. XII and Chap. XIV.