122. *Extension of Duhamel's Theorem.*

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Introduction. Regarding the conduction of heat we have the so-called Duhamel Theorem, which will at once give the solution when a solid initially at zero temperature, is exposed thereafter to any given *variable* temperature at the *surface,* if we know the solution for the solid whose surface is kept at a *constant* temperature other than the initial value.

Since Duhamel[1] first presented the theorem, the method of proof has been much improved by Riemann,[2] Carlslaw[3] and others, but they restricted themselves to heat-conduction corresponding to the differential equation $\frac{\partial\theta}{\partial t}=\nu\nabla^2\theta$ and the *boundary* temperature $\bar{\theta}=F(t)$.

In other branches of mathematical physics, however, no such general rule has been pronounced, and even special examples treated in similar manner are very rarely found, so far as the present writer knows.

The writer's new theorem here to be introduced, is a wide extension of Duhamel's, applicable not only to heat-conduction but also to several domains of physics and even pure mathematics. Moreover it may be used for the varying action or condition of *interior* bodily nature, as well as of boundary nature such as the surface temperature in Duhamel's theorem. The writer determines the limits within which the method similar to Duhamel theorem can be applied.

Definition. Let $x_1, x_2, \ldots\ldots, x_n$ be independent variables. Let a quantity $W(x_1, x_2, \ldots\ldots, x_n)$ be such a function as will be influenced by any other quantity $F(x_1, x_2, \ldots\ldots, x_n)$ which may be some "action" or "circumstance" in physical meaning.

Let $W_1, W_2, W_3, \ldots\ldots, W_m$ be the values of W corresponding to F's several values $F_1, F_2, F_3, \ldots\ldots, F_m$, provided that all other circumstances remain the same.

If, for the value of F

$$F=F_1+F_2+F_3+\cdots\cdots+F_m\,,$$

we have the corresponding value of W such that

$$W=W_1+W_2+W_3+\cdots\cdots+W_m\,,$$

then we define the quantity W to be *additive* or *superposable* with respect to F.

Theorem. Let a function $W(x_1, x_2, \ldots\ldots, x_n)$ be holomorphic for x_1 and additive with respect to $F(x_1, x_2, \ldots\ldots, x_n)$, and let

$$W=0\,,^{1)} \qquad F=0 \qquad \text{for} \qquad x_1<0\,.$$

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1) Journ. d. L'Ecole Polytech, **14**, 20–29 (1833).
2) Partielle Diff. Gleichungen II, 102–105 (1901).
3) The Conduction of Heat, 17 (1921).

If the solution of W for $x_1>0$, corresponding to an action constant with respect to x_1

$$F(\xi_1, x_2, x_3, \cdots\cdots, x_n)\,, \qquad \xi_1=\text{parameter}\,,$$

be known, viz.

$$W(\xi_1, x_1, x_2, x_3, \cdots\cdots, x_n)\,,$$

then the solution corresponding to an action variable with x_1

$$F(x_1, x_2, x_3, \cdots\cdots, x_n)$$

will be given by

$$W=\int_0^{x_1}\frac{\partial}{\partial x_1}W(\xi_1, x_1-\xi_1, x_2, x_3, \cdots\cdots, x_n)d\xi_1\,.$$

Proof. By assumption we know that if

$$\begin{aligned} F&=0 && \text{for} \quad -\infty<x_1<0 \\ &=\phi(\xi_1, x_2, x_3, \cdots\cdots, x_n) && \text{for} \quad 0\leqq x_1\,, \end{aligned}$$

the corresponding W is given by

$$W=\Psi(\xi_1, x_1, x_2, x_3, \cdots\cdots, x_n)\,.$$

Then if

$$\begin{aligned} F&=0 && \text{for} \quad x_1<\xi_1 \\ &=\phi(\xi_1, x_2, x_3, \cdots\cdots, x_n) && \text{for} \quad x_1\geqq\xi_1\,, \end{aligned}$$

we shall have

$$W=\Psi(\xi_1, x_1-\xi_1, x_2, x_3, \cdots\cdots. x_n) \qquad \text{for} \quad x_1>\xi_1\,,$$

since the condition is the same as before except only that the co-ordinate origin of x_1 is shifted by the amount ξ_1.

Similarly if

$$\begin{aligned} F&=0 && \text{for} \quad x_1<\xi_1+d\xi_1 \\ &=\phi(\xi_1, x_2, x_3, \cdots\cdots, x_n) && \text{for} \quad x_1\geqq\xi_1+d\xi_1\,, \end{aligned}$$

the corresponding W will be

$$W=\Psi(\xi_1, x_1-\xi_1-d\xi_1, x_2, x_3, \cdots\cdots, x_n) \qquad \text{for} \quad x_1>\xi_1+d\xi_1\,.$$

Hence if

$$\begin{aligned} F&=0 && \text{for} \quad x_1<\xi_1 \\ &=\phi(\xi_1, x_2, x_3, \cdots\cdots, x_n) && \text{from} \quad x_1=\xi_1 \quad \text{to} \quad x_1=\xi_1+d\xi_1 \\ &=0 && \text{for} \quad x>\xi_1+d\xi_1\,, \end{aligned}$$

the value of W for $x_1>\xi_1+d\xi_1$ will be given by

$$\begin{aligned} W&=\Psi(\xi_1, x_1-\xi_1, x_2, x_3, \cdots\cdots, x_n)-\Psi(\xi_1, x_1-\xi_1-d\xi_1, x_2, x_3, \cdots\cdots, x_n) \\ &=\frac{\partial}{\partial x_1}\Psi(\xi_1, x_1-\xi_1, x_2, x_3, \cdots\cdots, x_n)d\xi_1\,, \end{aligned}$$

remembering that the quantity W is *additive* with respect to F.

Thus, finally if the action is variable with x_1, i.e.,

$$F=\phi(x_1, x_2, x_3, \cdots\cdots, x_n)\,,$$

divide x_1 into many small intervals, and obtain corresponding elementary

1) If $W=C$ (a constant) for $x_1=0$, consider $W-C$ as a new W and test its additiveness.

influences and sum them up. The result will be

$$W=\int_0^{x_1}\frac{\partial}{\partial x_1}\Psi(\xi_1, x_1-\xi_1, x_2, x_3, \ldots\ldots, x_n)d\xi_1,$$

provided that W is *additive* with respect to F.

Physical examples. The fundamental equation specifying W may be a differential equation or an integral or of any other form; but if the equation is a linear differential equation, it will be very favorable for additiveness of W. In such cases, let us give a few examples in which the action F makes the quantity W additive.

(Ex. 1) *When the equation of W is a linear diff. eq. without second member of independent variables only.*

(a) *Case in which the boundary value of W is given as F:*—The ordinary Duhamel theorem for heat-conduction is an example of this kind.

(b) *Case where given F is a boundary surface-action proportional to a derivative of W:*—The writer's theory[1] of the drift-current in the ocean will afford an example of this sort. The eq. of motion of the current w is

$$\frac{\partial w}{\partial t}=\frac{\mu}{\rho}\frac{\partial^2 w}{\partial z^2}-2i\bar{\omega}w, \qquad i=\sqrt{-1},$$

with the conditions

$$w=0 \quad \text{at} \quad t=0$$

and

$$\partial w/\partial z=-iT/\mu \quad \text{at} \quad z=0,$$

where T is the tangential stress of the wind.

Since such current w is obviously additive with respect to T, the present theorem may be applied, and the writer actually determined the current for variable T in that way, the result being coincident with Fjeldstad's solution[2] obtained by another method.

Kelvin's theory[3] of the electric telegraph determines the mode of electric transmission when the key is put at momentary contact, but we can evidently extend it to a more general case by the present theorem, (a) or (b).

(Ex. 2) *When F is an internal bodily action and is given as the second member of the linear diff. eq. of W.*

The writer's theory[4] of slope- or barometric current in the ocean will afford examples. The equation of motion of the current due to a slope γ is

$$\frac{\partial w}{\partial t}=\frac{\mu}{\rho}\frac{\partial^2 w}{\partial z^2}-2i\bar{\omega}w+ig\gamma.$$

The solution of w when γ varies with time t was obtained from the solution for constant γ. The writer used the theorem in his theory of tunamis[5] also.

Another example will be found in Proudman's formula[6] (2.71) given in his paper on "The effects on the sea of changes in atmospheric pressure."

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1) Mem. Coll. Sci. Kyoto Imp. Univ., A, **16**, 171 and 287 (1933).
2) Zeits. f. angew, Math. u. Mech. **10**, 121 (1930).
3) Math. & Phys. Papers, II (1884), p. 61.
4) Mem. Coll. Sci. Kyoto Imp. Univ., A, **16**, 213 & 336 (1933); **17**, 274 (1934).
5) Ditto, **18**, 211 (1935).
6) M. N. R. A. S. Geophys. Suppl., **2**, 197 (1929).