PAPERS COMMUNICATED

103. *A Theorem on the Conjugate Functions.*

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I will, in the present paper, show that a function and its conjugate function can not be both very small at $\pm\infty$. Or I will prove the following theorem.

Theorem. Let $p(x)$ be defined for $x>0$ and positive and be such that $e^{-p(x)}$ is squarely integrable in $(-\infty, \infty)$. Then in order that there should exist a non-null function $f(x)$ defined in $(-\infty, \infty)$ such that $f(x)$ and its conjugate function $\bar{f}(x)$ are both in absolute values smaller than $Ae^{-p(|x|)}$ almost everywhere, where A is a constant independent of x, it is necessary and sufficient that

$$\int_0^\infty \frac{p(x)}{1+x^2}\,dx$$

should be convergent.

We can prove this theorem by combining the Paley and Wiener's fundamental theorem in the theory of quasi-analytic functions[1] and a theorem due to E. C. Titchmarsh which can be stated as follows:

If $f(x)$ is squarely integrable and $F(x)$ is its Fourier transform, then the Fourier transform of the conjugate function of $f(x)$ is $-iF(x)\,\mathrm{sgn}\,x$.[2]

Necessity. Let $f(x)$ be non-null and be such that

$$(1)\qquad |f(x)|\leqq Ae^{-p(|x|)}, \qquad |\bar{f}(x)|\leqq Ae^{-p(|x|)}.$$

If $F(x)$ denotes the Fourier transform of $f(x)$, then, by the Titchmarsh's theorem, the Fourier transform of $\bar{f}(x)$ is $-iF(x)\,\mathrm{sgn}\,x$. Clearly $F(x)$ is non-null and we suppose that it is not null for $x>0$, otherewise we can proceed similarly. Then we have

$$f(x)+i\bar{f}(x)\sim\sqrt{\frac{2}{\pi}}\int_0^\infty F(t)e^{-itx}dt=\sqrt{\frac{2}{\pi}}\int_{-\infty}^0 F(-t)e^{itx}dt.$$

Or $f(x)+i\bar{f}(x)$ is the Fourier transform of a function vanishing for positive arguments. Hence we have, by the Paley and Wiener's theorem,

$$(2)\qquad \int_{-\infty}^\infty \frac{|\log|f(x)+i\bar{f}(x)||}{1+x^2}\,dx<\infty.$$

From (1) we have

$$p(|x|)\leqq 2|\log A|+|\log|f(x)+i\bar{f}(x)||.$$

1) Paley and Wiener, Fourier transforms in the complex domain, Amer. Math. Soc. Colloqium.

2) Titchmarsh, Conjugate trigonometrical integrals, Proc. London Math. Soc. **24** (1925).

Thus from (2) we reach the result:

$$\int_{-\infty}^{\infty} \frac{p(|x|)}{1+x^2}\, dx < \infty .$$

Sufficiency. Let $\int_{-\infty}^{\infty} \frac{p(|x|)}{1+x^2}\, dx < \infty$. Then by the Paley and Wiener's theorem there exists a non-null function $f(x)$ such that $f(x)=0$ for $x > x_0$ for some x_0 and its Fourier transform $F(x)$ satisfies

$$e^{-p(|x|)} = |F(x)| .$$

We can suppose that $x_0=0$, for otherwise we consider the function $f(x-x_0)$, its Fourier transform being in the absolute value equal to $|F(x)|$. Let the conjugate function of $F(x)$ be $\bar{F}(x)$. By the Titchmarsh's theorem, we can see that $\bar{F}(x)$ is the Fourier transform of $-if(x)\operatorname{sgn} x$ which is really $if(x)$, since $f(x)$ is zero for $x > 0$. Thus $|\bar{F}(x)| = |F(x)|$. Thus the theorem is proved.