43. A Problem Concerning the Second Fundamental Theorem of Lie.

By Kôsaku YOSIDA.

Mathematical Institute, Faculty of Science, Osaka Imperial University.

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§ 1. The problem and the theorem.

Let $\mathfrak{R}$ denote the set of all the matrices of a fixed degree, say n, with complex numbers as coefficients. We introduce a topology in $\mathfrak{R}$ by the *absolute value*

$$|A| = \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2}, \qquad A = \| a_{ij} \| .$$

If $\mathfrak{G}$, a subset of non-singular matrices $\in \mathfrak{R}$, is a group with respect to the matrix-multiplication, it is a topological group by the distance $|A-B|$.

The topological group $\mathfrak{G}$ is called a *Lie group*, if there exist a finite number, say m, of elements $X_1, X_2, \ldots\ldots, X_m \in \mathfrak{R}$ which satisfy the conditions:

1). $X_1, X_2, \ldots\ldots, X_m$ are linearly independent with real coefficients.

2). $\exp(\sum_{i=1}^{m} t_i X_i) \in \mathfrak{G}$, t real.[1)]

3). There exists a positive ε such that any element $A \in \mathfrak{G}$ may be represented uniquely in the form

$$A = \exp(\sum_{i=1}^{m} t_i X_i), \ t \text{ real},$$

if $|A-E| \leqq \varepsilon$ (E the unit-matrix of $\mathfrak{R}$).

By a theorem of J. von Neumann[2)] $\mathfrak{G}$ is a Lie group if and only if it is locally compact. Here, for convention, a discrete group is also called a Lie group. If $\mathfrak{G}$ is a Lie group, the set $\mathfrak{J}$ of all the elements $\sum_{i=1}^{m} t_i X_i$, t real, satisfies:

(α). $\mathfrak{J}$ is a real linear space which has a finite base with real coefficients, viz, $X_1, X_2, \ldots\ldots, X_m$.

(β). $[X, Y] = XY - YX \in \mathfrak{J}$ with $X, Y \in \mathfrak{J}$.

$\mathfrak{J}$ is called the *Lie ring* of the Lie group $\mathfrak{G}$, the two ring-operations being the vector-addition and the *commutator-multiplication* $[X, Y]$. It is the set of all the *differential quotients* of $\mathfrak{G}$ at E.[3)] The differential quotient of $\mathfrak{G}$ at E is defined by $\lim_{i\to\infty} \big((A_i - E)/\varepsilon_i\big)$, where $A_i (\neq E) \in \mathfrak{G}$ and real ε_i $(\neq 0)$ are such that $\lim_{i\to\infty} A_i = E$, $\lim_{i\to\infty} \varepsilon_i = 0$.

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1) $\exp(X) = \sum_{n=0}^{\infty} (X^n/n!)$.

2) See K. Yosida: Jap. J. of Math. **13** (1936), p. 7. Neumann's original statement (M. Z. **30** (1929), p. 3) reads as follows:

$\mathfrak{G}$ is a Lie group if $\mathfrak{G}$ is closed in the group of all the non-singular matrices $\in \mathfrak{R}$.

3) Cf. K. Yosida: loc. cit.

Conversely let $\bar{\mathfrak{J}}$ denote a subset of $\mathfrak{R}$ which satisfies (α) and (β). Then, by the second fundamental theorem of Lie, the set $\bar{\mathfrak{G}}$ of all the elements of the form

$$\exp\,(\sum_{i=1}^{m} t_i X_i)\,,\ t \text{ real and } \sum_{i=1}^{m} |\,t_i\,| < \varepsilon\,, \qquad \varepsilon > 0\,,$$

constitutes a *Lie group-germ*. That is, if $X, Y \in \bar{\mathfrak{G}}$ are sufficiently near E, X^{-1} and YX also $\in \bar{\mathfrak{G}}$. $\bar{\mathfrak{J}}$ is called the Lie ring of the Lie group-germ $\bar{\mathfrak{G}}$.

Then the set $\tilde{\mathfrak{G}}$ of all the products of a finite number of elements $\in \bar{\mathfrak{G}}$ and of the limit matrices of such products, so long as they are non-singular, forms a locally compact group. Hence $\tilde{\mathfrak{G}}$ is a Lie group. Let $\tilde{\mathfrak{J}}$ be the Lie ring of this Lie group $\tilde{\mathfrak{G}}$, then $\tilde{\mathfrak{J}} \geqq \bar{\mathfrak{J}}$. However, $\tilde{\mathfrak{J}}$ does not necessarily coincide with $\bar{\mathfrak{J}}$, as the following example shows us:

$$\text{the base of } \bar{\mathfrak{J}} = \begin{Vmatrix} \sqrt{-1} & 0 \\ 0 & \tau\sqrt{-1} \end{Vmatrix}, \quad \tau/2\pi \text{ irrational.}$$

Hence the Lie group-germ $\bar{\mathfrak{G}}$ is not necessarily a vicinity of the identity of the topological group $\tilde{\mathfrak{G}}$.

Thus it may be of some interest to obtain the conditions by which $\bar{\mathfrak{J}}$ coincides with $\tilde{\mathfrak{J}}$. As an answer to this problem, I intend to prove the following

Theorem. The Lie group-germ $\bar{\mathfrak{G}}$ is a vicinity of the identity of the Lie group $\tilde{\mathfrak{G}}$, if the ring $\bar{\mathfrak{J}}$ is irreducible.

Here $\bar{\mathfrak{J}}$ is called *irreducible* if the group $\tilde{\mathfrak{G}}$ is irreducible, that is, if all the matrices of $\bar{\mathfrak{J}}$ are not simultaneously similar to the matrices of the form

$$\begin{Vmatrix} A & 0 \\ * & B \end{Vmatrix}$$

§ 2. The proof of the theorem.

Lemma 1. $\bar{\mathfrak{G}}$ is a Lie invariant subgroup-germ of $\tilde{\mathfrak{G}}$, viz. $BAB^{-1} \in \bar{\mathfrak{G}}$ for any $B \in \tilde{\mathfrak{G}}$ if $A \in \bar{\mathfrak{G}}$ is sufficiently near E.

Proof. Let $A = \exp(X)$, $X \in \bar{\mathfrak{J}}$. Then $BAB^{-1} = \exp(BXB^{-1})$ and BXB^{-1} tends to 0 as X tends to 0. Thus it is sufficient to prove

$$(*) \qquad BXB^{-1} \in \bar{\mathfrak{J}} \text{ with } X \in \bar{\mathfrak{J}}\,, \qquad \text{if } B \in \tilde{\mathfrak{G}}\,.$$

(*) is evident in the special case $B \in \bar{\mathfrak{G}}$, for then the transformation $X \to BXB^{-1}$ is induced by the so-called linear adjoint Lie group-germ of $\bar{\mathfrak{G}}$. The general case $B \in \tilde{\mathfrak{G}}$ may be obtained from this special case, by limiting process.

Lemma 2 (due to E. Cartan[1]). The vicinity of the identity of the irreducible Lie group $\tilde{\mathfrak{G}}$ is a direct product of a *semi-simple* Lie

1) E. Cartan: Ann. Ec. Norm. Sup. (3) **26** (1909), p. 148. For the proof see H. Freudenthal: Ann. of Math. 37, **1** (1936), p. 63. In the course of the proof of our theorem, $\bar{\mathfrak{G}}_i$ ($i=1, 2$) will be proved to be not only Lie group-germ but also a vicinity of the identity of the Lie group.

group-germ $\bar{\mathfrak{G}}_1$ and an abelian Lie group-germ $\bar{\mathfrak{G}}_2$, where det. $(A)=1$ for any $A \in \bar{\mathfrak{G}}_1$ and the matrices of $\bar{\mathfrak{G}}_2$ are all of the form aE, a denoting complex numbers.

As a special case of this Lemma we have

Lemma 2′. $\tilde{\mathfrak{G}}$ is a semi-simple Lie group if $\bar{\mathfrak{J}}$ is irreducible and

$$(**) \qquad \text{trace}\,(X)=0 \qquad \text{for } X \in \bar{\mathfrak{J}}.$$

Proof. For then the matrices of $\bar{\mathfrak{G}}$ and hence of $\tilde{\mathfrak{G}}$ are all of determinant 1.[1]

The above condition (**) is surely satisfied if the Lie ring $\bar{\mathfrak{J}}$ is semi-simple. For a semi-simple Lie ring $\bar{\mathfrak{J}}$ coincides with its *commutator-ring*,[2] that is, any element of $\bar{\mathfrak{J}}$ may be obtained as the commutator-product $[X, Y]$, where X and $Y \in \bar{\mathfrak{J}}$.

Proof of the theorem. By Lemma 1 the sub-ring $\bar{\mathfrak{J}}$ is an *ideal* in $\tilde{\mathfrak{J}}$, viz. $[X, Y] \in \bar{\mathfrak{J}}$ for $X \in \bar{\mathfrak{J}}$, $Y \in \tilde{\mathfrak{J}}$. We will prove that this ideal $\bar{\mathfrak{J}}$ is a direct summand of the Lie ring $\tilde{\mathfrak{J}}$.

By Lemma 2 the Lie ring $\tilde{\mathfrak{J}}$ is a direct sum of the semi-simple Lie ring $\tilde{\mathfrak{J}}_1$ of the Lie group-germ $\bar{\mathfrak{G}}_1$ and the abelian Lie ring $\tilde{\mathfrak{J}}_2$ of the Lie group-germ $\bar{\mathfrak{G}}_2$. Thus $\tilde{\mathfrak{J}}_1$ is commutative with $\tilde{\mathfrak{J}}_2$: $[X, Y]=0$ for $X \in \tilde{\mathfrak{J}}_1$, $Y \in \tilde{\mathfrak{J}}_2$.

The semi-simple Lie ring $\tilde{\mathfrak{J}}_1$ is a direct sum of simple and semi-simple ideals, by a theorem of E. Cartan.[3] Hence any ideal of $\tilde{\mathfrak{J}}_1$ is semi-simple. As $\bar{\mathfrak{G}}_2$ consists of the matrices of the form aE, the base of the abelian Lie ring $\tilde{\mathfrak{J}}_2$ is either

i). aE, where a denotes a real or complex number ($a=0$ if $\tilde{\mathfrak{J}}_2=0$),

or

ii). E and $\sqrt{-1}E$.

Thus, in any case, $\tilde{\mathfrak{J}}$ is a direct sum of simple ideals. Hence the ideal $\bar{\mathfrak{J}}$ is a direct summand of $\tilde{\mathfrak{J}}$. We next prove that $\bar{\mathfrak{J}} \geqq \tilde{\mathfrak{J}}_1$.

Let $\tilde{\mathfrak{J}}=\bar{\mathfrak{J}}+\bar{\mathfrak{J}}'$ be a direct decomposition of $\tilde{\mathfrak{J}}$. Then, as $\bar{\mathfrak{J}}$ and $\bar{\mathfrak{J}}'$ are ideals in $\tilde{\mathfrak{J}}$, $\bar{\mathfrak{J}}$ is commutative with $\bar{\mathfrak{J}}'$:

$$(***) \qquad [X, Y]=0 \qquad \text{for } X \in \bar{\mathfrak{J}}, \qquad Y \in \bar{\mathfrak{J}}'.$$

Hence, if $\bar{\mathfrak{J}}$ does not contain $\tilde{\mathfrak{J}}_1$, there must exist a semi-simple ideal $\tilde{\mathfrak{J}}_1' \leqq \tilde{\mathfrak{J}}_1$, commutative with $\bar{\mathfrak{J}}$ by (***). Thus the matrices $\in \tilde{\mathfrak{G}}$ of the form $\exp(X)$, $X \in \tilde{\mathfrak{J}}_1'$, are permutable with every matrix of the irreducible group-germ $\bar{\mathfrak{G}}$. Hence, by Schur's Lemma, $\exp(X)$ $(X \in \tilde{\mathfrak{J}}_1')$ and consequently every matrix $\in \tilde{\mathfrak{J}}_1'$ must be of the form aE. $\tilde{\mathfrak{J}}_1'$ is thus an abelian Lie ring and hence is not semi-simple. This is a contradiction, and so we must have $\bar{\mathfrak{J}} \geqq \tilde{\mathfrak{J}}_1$.

The same reasoning shows that, if $\tilde{\mathfrak{J}}$ is irreducible and semi-simple, we must have $\bar{\mathfrak{J}}=\tilde{\mathfrak{J}}$. For, then $\tilde{\mathfrak{J}}$ is semi-simple by Lemma 2′. Hence, in the above Lemma 2, *$\bar{\mathfrak{G}}_1$ and $\bar{\mathfrak{G}}_2$ are not only Lie group-germ but also the vicinities of the identities of Lie groups.*

Next we will prove that $\bar{\mathfrak{J}} \geqq \tilde{\mathfrak{J}}_2$. There are two cases.

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1) det. $(\exp(X))=\exp(\text{trace}\,(X))$.

2) See, for example, H. Freudenthal: loc. cit.

3) E. Cartan: Thèses (1894), p. 53.

Case 1. Base of $\bar{\mathfrak{J}}_2 = aE (a=0$ if $\bar{\mathfrak{J}}_2 = 0)$.

Assume that $\bar{\mathfrak{J}}_2 \neq 0$ and $\bar{\mathfrak{J}} = \bar{\mathfrak{J}}_1$. Then the group-germ $\bar{\mathfrak{G}}$ is a vicinity of the identity of the Lie group $\tilde{\mathfrak{G}}_1$ whose Lie ring are $\bar{\mathfrak{J}}_1 = \bar{\mathfrak{J}}$. Thus $\bar{\mathfrak{J}}_2 = 0$, contrary to the hypothesis. This proves $\bar{\mathfrak{J}} \geqq \bar{\mathfrak{J}}_2$.

Case 2. Base of $\bar{\mathfrak{J}}_2 = E$ and $\sqrt{-1}E$.

If both E and $\sqrt{-1}E$ do not belong to $\bar{\mathfrak{J}}$, we obtain $\bar{\mathfrak{J}}_2 = 0$ as above, contrary to the hypothesis $\bar{\mathfrak{J}}_2 \neq 0$. Next let either one of E and $\sqrt{-1}E$, E for example, belong to $\bar{\mathfrak{J}}$. Then, as E is permutable with every matrix, any matrix $\in \tilde{\mathfrak{G}}$ must be of the form

$$A_1 A_2 \cdots\cdots A_k Y, \quad \text{where} \begin{cases} A_i \in \text{the intersection } (\bar{\mathfrak{G}} \cdot \bar{\mathfrak{G}}_1), \\ Y = \exp(tE), \ t \text{ real}, \end{cases}$$

or the limit matrix of such matrices. Thus, by Lemma 2, det. $(X) = \exp(t)$, t real, for $X \in \tilde{\mathfrak{G}}$, and hence X is not of the form $\exp(s \cdot \sqrt{-1}E)$, s real. Then $\sqrt{-1}E$ does not belong to the Lie ring $\tilde{\mathfrak{J}}$. This is a contradiction, and so we must have $\bar{\mathfrak{J}} \geqq \bar{\mathfrak{J}}_2$.

Thus, in any case, $\bar{\mathfrak{J}} = \tilde{\mathfrak{J}}$.

Q. E. D.