12. *On a Theorem of K. Löwner on Univalent Functions.*

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1. We denote by (S) the family of normalized univalent functions

$$(1) \qquad f(z)=z+a_2z^2+a_3z^3+\cdots\cdots,$$

regular for $|z|<1$, and by (S_k), $k=2, 3, \ldots\ldots$, the family of normalized univalent functions given by

$$(2) \qquad f_k(z)=f^{\frac{1}{k}}(z^k)=z+a_{k+1}^{(k)}z^{k+1}+a_{2k+1}^{(k)}z^{2k+1}+\cdots\cdots.$$

Löwner[1] succeeded to prove that, for any function of (S),

$$(3) \qquad |a_3|\leqq 3,$$

which relates to the Bieberbach's conjecture, using his own essential theorem of expressing all coefficients of any function $f(z)$ of (S) by parametric function corresponding to $f(z)$ itself.

On the other hand, Fekete and Szegö[2] have recently proved that, for $k=2, 3, \ldots\ldots$,

$$(4) \qquad |a_{2k+1}^{(k)}|\leqq \frac{2}{k}e^{-2[(k-1)/(k+1)]}+\frac{1}{k}.$$

It is remarkable for us that the inequality (4) becomes for $k=2$,

$$(5) \qquad |a_5^{(2)}|\leqq e^{-\frac{2}{3}}+\frac{1}{2}>1,$$

and the extremal case of (4) and (5) are attained by a function which is different from the well known extremal function

$$\frac{z}{(1-z^k)^{2/k}}, \qquad (k=2, 3, \ldots\ldots),$$

by which the Bieberbach's conjecture for the class of functions (S) is realized for $k=1$.

In the present note, a coefficient problem closely related to the above mentioned theorems of Löwner, Fekete and Szegö is investigated.

2. Instead of the class (S), Löwner considered the class (S') of univalent functions of the form

$$(6) \qquad s(z, t)=e^{-t}\left(z+b_1(t)z^3+b_2(t)z^3+\cdots\cdots\right), \qquad t\geqq 0,$$

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1) Math. Annalen, **89** (1923), 103-121. .
2) Jour. London Math. Soc., **8** (1933), 85-89.

regular for $|z| \leqq 1$ under the condition that $|s(z,t)| \leqq 1$, $s(z,o)=z$ and coefficients $b_1(t)$, $b_2(t)$, ... are functions of t.

By the theory of Löwner, it is true that from any function $s(z,t)$ of (S') we get in the limiting case of $t \to \infty$,

$$\lim_{t\to\infty} \frac{s(z,t)}{e^{-t}} = \lim_{t\to\infty}\left(z+b_1(t)z^2+b_2(t)z^3+\cdots\cdots\right) \tag{7}$$
$$= z+a_2z^2+a_3z^3+\cdots\cdots,$$

which is a function of (S).

Let

$$s(z,t_0)=e^{-t_0}\left(z+b_1(t_0)z^2+b_2(t_0)z^3+\cdots\cdots\right)$$

be any function of (S'), then Löwner's expression for $b_1(t_0)$, $b_2(t_0)$ are given by

$$\begin{cases} b_1(t_0)=-2\int_0^{t_0} \varkappa(\tau)e^{-\tau}d\tau\,, \\ b_2(t_0)=4\left[\int_0^{t_0} \varkappa(\tau)e^{-\tau}d\tau\right]^2-2\int_0^{t_0} \varkappa^2(\tau)e^{-2\tau}d\tau\,, \end{cases} \tag{8}$$

where $\varkappa(t)$ is the parametric function of the function $s(z,t_0)$ such that $|\varkappa(t)|=1$ and is continuous but discontinuous at atmost finite points in the interval $0 \leqq t \leqq t_0$.

Löwner's results obtained from (8) are

$$|b_1(t_0)| \leqq 2(1-e^{-t_0})\,, \tag{9}$$

$$|b_2(t_0)| < 3-4e^{-t_0}+e^{-2t_0}\,. \tag{10}$$

The inequality (9) is the best and identical with that of Pick[1] and its extremal case is attained by the function $s(z)$ given by the equation

$$\frac{s(z)}{\left(1+\varepsilon s(z)\right)^2}=e^{-t_0}\frac{z}{(1+\varepsilon z)^2}\,, \qquad |\varepsilon|=1\,. \tag{11}$$

And the inequality (10) is not the best, but for the limit $t_0 \to \infty$ it becomes to the best inequality $|a_3| \leqq 3$ for the class (S).

3. In the following we get the best inequality for $|b_2(t_0)|$ instead of (10) and prove that the exact maximum of $|b_2(t_0)|$ for any fixed t_0 ($t_0 \geqq 1$) is not attained by the corresponding coefficient of the extremal function given by (11), in which we can see in the class of functions (S), (S'), there exists the close relation to the theorem of Fekete-Szegö for (S_k), $k=2, 3, \cdots\cdots$.

Theorem 1. For the functions of the class (S') for $t=t_0$,

$$\operatorname*{Max}_{(S')}|b_2(t_0)| \leqq 4\Big\{(\nu_0e^{-\nu_0}+e^{-\nu_0}-e^{-t_0})^2 \tag{12}$$
$$-\left(\nu_0e^{-2\nu_0}+\frac{1}{2}e^{-2\nu_0}-\frac{1}{2}e^{-2t_0}\right)\Big\}+1-e^{-2t_0}\,,$$

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1) Wien. Ber. **126**, Abtlg. (2) a (1917), 247-263.

where $\nu_0=\nu_0(t_0)$, $(0 \leqq \nu_0 \leqq t_0)$, is given by the root of the equation

$$\nu_0 e^{-\nu_0}=e^{-t_0}. \tag{13}$$

We see that

$$\lim_{t_0\to\infty} \nu_0(t_0)=0,$$

and for $t_0 \to \infty$ (12) becomes

$$\operatorname*{Max}_{(S)} |a_3| \leqq 3,$$

which is consistent with the Löwner's theorem and inequality (12) is the best for $t_0 \geqq 1$.

To prove the theorem 1, we must use the following

Theorem 2. Let $\lambda(\tau)$ be a function, real and continuous except atmost one point in the interval $0 \leqq \tau \leqq t_0$, and be

$$|\lambda(\tau)| \leqq e^{-\tau},$$

and also for N such that

$$0 < N \leqq \frac{1}{2}(1-e^{-2t_0}),$$

$$\int_0^{t_0} \lambda^2(\tau)\, d\tau \leqq N.$$

Then

$$\left| \int_0^{t_0} \lambda(\tau)\, d\tau \right| \leqq V(N), \tag{14}$$

where $V(N)=(\nu+1)e^{-\nu}-e^{-t_0}$ and $\nu=\nu(N)$, $(0 \leqq \nu \leqq t_0)$, is the root of the equation

$$\left(\nu+\frac{1}{2}\right)e^{-2\nu}-\frac{1}{2}e^{-2t_0}=N \qquad \text{when} \quad t_0 e^{-2t_0} \leqq N \leqq \frac{1}{2}(1-e^{-2t_0}),$$

$$\text{and} \quad \nu(N)=t_0 \qquad \text{when} \quad 0 < N \leqq t_0 e^{-2t_0}.$$

Applying theorem 2 we arrive at the following Valiron-Landau's theorem[1] in a generalized form.

Theorem 3. Let $\kappa(\tau)=e^{i\vartheta(\tau)}$ be a function, continuous except atmost one point in the interval $0 \leqq \tau \leqq t_0$, and be

$$|\kappa(\tau)|=1,$$

and also for M such that

$$0 < M \leqq \frac{1}{2}(1-e^{-2t_0}),$$

$$\left| \int_0^{t_0} \kappa^2(\tau) e^{-2\tau} d\tau \right| \leqq M.$$

Then

$$\left| \int_0^{t_0} \kappa(\tau) e^{-\tau} d\tau \right| \leqq V\left(\frac{1}{4}+\frac{M}{2}-\frac{1}{4}e^{-2t_0}\right), \tag{15}$$

1) Math. Zeitschr., **30** (1929), 608–634.

and we can see that for $t_0 \geqq 1$, there exists in fact a function $\varkappa(\tau)=e^{i\vartheta(\tau)}$ which realize the equality in (15).

4. Now using theorem 3, the proof of our theorem 1 is immediately given.

$$b_2(t_0)=4\left[\int_0^{t_0}\varkappa(\tau)e^{-\tau}d\tau\right]^2-2\int_0^{t_0}\varkappa^2(\tau)e^{-2\tau}d\tau\,,$$

and here, without loss of generality, we can assume that $|b_2(t_0)|=\Re b_2(t_0)$, because in the other case we replace $\varkappa(\tau)$ by $\varepsilon\varkappa(\tau)$, $|\varepsilon|=1$, and using theorem 3,

$$\begin{aligned}|b_2(t_0)|=\Re b_2(t_0)&=4\left\{\left(\int_0^{t_0}\cos\vartheta(\tau)\cdot e^{-\tau}d\tau\right)^2-\left(\int_0^{t_0}\sin\vartheta(\tau)\cdot e^{-\tau}d\tau\right)^2\right.\\&\qquad\left.-\int_0^{t_0}\cos^2\vartheta(\tau)\cdot e^{-2\tau}d\tau\right\}+1-e^{-2t_0}\\&\leqq 4\left\{\left(\int_0^{t_0}\cos\vartheta(\tau)\cdot e^{-\tau}d\tau\right)^2-\int_0^{t_0}\cos^2\vartheta(\tau)\cdot e^{-2\tau}d\tau\right\}+1-e^{-2t_0}\\&\leqq 4\left\{\left(\nu e^{-\nu}+e^{-\nu}-e^{-t_0}\right)^2-\left(\nu e^{-2\nu}+\frac{1}{2}e^{-2\nu}-\frac{1}{2}e^{-2t_0}\right)\right\}\\&\qquad+1-e^{-2t_0}\equiv F(\nu,t_0)\,,\qquad\text{put.}\end{aligned}$$

The maximum of $F(\nu, t_0)$ is given when $\nu=\nu_0$ $(0\leqq\nu_0\leqq t_0)$ which is given by the root of the equation

$$\nu e^{-\nu}=e^{-t_0}\,, \tag{16}$$

and we see that

$$F(\nu_0, t_0)>3-8e^{-t_0}+5e^{-2t_0}=F(0, t_0)\quad\text{for}\quad t_0\geqq 1\,.$$

Thus we get the proof of our theorem 1. The details of this and other relating problems shall be published soon in the another place.