68. *Remarks on the Representation of Entire Functions of Exponential Type.*

By Tatsuo KAWATA.

Mathematical Institute, Tohoku Imperial University, Sendai.

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1. Paley and Wiener proved in their work "Fourier transforms in the complex domain, Amer. Math. Colloq. XIX" that the entire function $F(z)$ can be represented as

$$F(z)=\int_{-A}^{A} e^{izt}f(t)at \tag{1.1}$$

with $f(t)$ of L_2[1], if and only if $F(z)$ satisfies

$$F(z)=O\,(e^{A|z|}) \tag{1.2}$$

and belongs to L_2 on the real axis. Extending to L_p class Plancherel-Pólya[2] and R. P. Boas[3] have proved the following theorems:

I. *If the entire function $F(z)$ satisfies* (1.2) *and belongs to L_p $(1<p\leqq 2)$ on the real axis, then $F(z)$ has the representation* (1.1) *with $f(t)$ of $L_{p'}(-A, A)$, p' being conjugate to p.*

II. *If* (1.1) *holds with $f(t)\in L_p(1<p\leqq 2)$, then $F(z)$ is entire and satisfies* (1.2) *and belongs to $L_{p'}$ on the real axis.*

The object of the present paper is to show that we can prove the theorems in the further generalized forms by the analogous method as original Paley and Wiener's one and using some results due to Hille and Tamarkin.

2. Let $F(x)$ belong to $L_p(1\leqq p\leqq\infty)$. If there exists a function $f(x)$ of L_q $(1\leqq q<\infty)$ such that

$$\lim_{N\to\infty}\int_{-\infty}^{\infty}\left|\frac{1}{\sqrt{2\pi}}\int_{-N}^{N}F(x)e^{-iux}dx-f(u)\right|^{q}du=0\,, \tag{2.1}$$

then we call $f(x)$ the Fourier transform in L_q of $F(x)$. By the theory of Fourier transform, if $F(x)\in L_p(1<p\leqq 2)$, then $F(x)$ has the Fourier transform in $L_{p'}$ and $F(x)$ itself is the Fourier transform in L_p of some function in $L_{p'}$.

Thus the following theorems are generalizations of I and II.

Theorem 1. If the entire function $F(z)$ satisfies (1.2) *and it be-*

1) L_p means $L_p(-\infty,\infty)$ otherwise specified.

2) Plancherel and Pólya proved actually the theorem in the case of a function of many variables. Commentarii Helv. Mat. **9** (1936–37).

3) R. P. Boas, Jr. Representations for entire functions of exponential type. Annals of Math. **39** (1938), No. 2.

Hua and Shü obtained the corresponding theorems for $f(z)$ of $H^pL^p(p>2)$ on the real axis. The class H_pL_p is the one introduced by A. C. Offord.

Cf. A. C. Offord, On Fourier transforms III. Transact. Amer. Math. Soc. Hua and Shü, On Fourier transforms in L_p in the complex domain. Journal of Math. and Phys. **15** (1936).

longs to $L_p(1<p<\infty)$ *and has the Fourier transform in* $L_q(1<q<\infty)$ *on the real axis, then* $f(z)$ *has the representation* (1.1) *with* $f(t)\in L_q$ $(-A, A)$.

Theorem 2. In the preceding theorem the condition that on the real axis $F(x)$ *has the Fourier transform in* L_q, *can be replaced by the condition that* $F(x)$ *itself is the Fourier transform in* $L_p(1<p<\infty)$ *of some function of* L_q.

Theorem 3. If $f(x)\in L_p$ *and has the Fourier transform in* L_q $(1<q<\infty)$, *then*

$$F(z)=\int_{-A}^{A} f(x)e^{izx}dx$$

is the entire function satisfying (1.2) *and belongs to* L_q *on the real axis.*

Theorem 4. Theorem 3 is also true if $f(x)$ *satisfies the condition that it is the Fourier transform in* $L_p(1<p<\infty)$ *of some function of* $L_q(1<q<\infty)$.

3. We now proceed to prove Theorem 1. Since $F(z)=O(e^{A|z|})$,

$$F(z)=o(e^{B|z|}), \qquad B>A. \tag{3.1}$$

Consider the function

$$G(z)=\frac{e^{iBz}}{\varepsilon}\int_{z}^{z+\varepsilon} F(w)dw$$

which is clearly entire and satisfies

$$|G(z)|=o(e^{(A+B)|z|}), \tag{3.2}$$

thus $G(z)$ is of exponential type. And

$$G(iy)=\frac{e^{-By}}{\varepsilon}\int_{iy}^{iy+\varepsilon} F(w)\,dw=\frac{e^{-By}}{\varepsilon}\int_{0}^{\varepsilon} F(iy+x)dx=o(1), \tag{3.3}$$

$$|G(x)|\leqq\frac{1}{\varepsilon^{1/p}}\left(\int_{x}^{x+\varepsilon}|F(t)|^p dt\right)^{\frac{1}{p}}\leqq\frac{1}{\varepsilon^{1/p}}\left(\int_{-\infty}^{\infty}|F(t)|^p dt\right)^{\frac{1}{p}}=O(1). \tag{3.4}$$

Hence by the well known theorem of Phragmén-Lindelöf, $G(z)$ is bounded in $\Im(z)\geqq 0$. Further we get

$$\int_{-\infty}^{\infty}|G(x)|^p dx=\int_{-\infty}^{\infty}\left|\frac{1}{\varepsilon}\int_{x}^{x+\varepsilon}F(t)dt\right|^p dx$$

which does not exceed, by Hölder's inequality $\int_{-\infty}^{\infty}|F(t)|^p dt$ and thus $G(z)$ belongs to L_p on the real axis. From this fact and the boundedness of $G(z)$ in $\Im(z)\geqq 0$, we can represent as

$$G(z)=\text{const.}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{G(x)}{x-z}dx, \qquad (\Im(z)>0). \tag{3.5}$$

This can be proved by the completely analogous method as in Paley-Wiener's book.[1)]

1) Paley-Wiener, loc. cit. pp. 10–11.

By (3.3), the constant in (3.5) is zero. Thus $G(z)$ is represented by its Cauchy integral. We use now the result of Hille and Tamarkin[1] that if the analytic function in $\Im(z)>0$ has the boundary function belonging to L_p on the real axis and is represented as its Cauchy integral, then it is also represented as Poisson integral of its boundary function.

This theorem shows that

$$G(z)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{yG(\xi)}{(\xi-x)^2+y^2}d\xi\,, \qquad (z=x+iy)\,. \tag{3.6}$$

Now we shall show that with $F(x)$, $G(x)$ has the Fourier transform in L_q. Let $f(x)\in L_q$ be the Fourier transform of $F(x)$ and let $\Phi(x)$ be the Fourier transform of $\varphi(x)$ which is defined as

$$\varphi(x)=\frac{1}{\varepsilon}\,, \qquad 0<x<\varepsilon\,,$$

$$=0\,, \qquad \text{otherwise.}$$

Then

$$G(x,N)=\frac{1}{\sqrt{2\pi}}\int_{-N}^{N}G(u)e^{-iux}du$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-N}^{N}e^{-i(x-B)u}du\int_{-\infty}^{\infty}F(\xi)\varphi(\xi-u)d\xi$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-N}^{N}e^{-i(x-B)u}du\int_{-\infty}^{\infty}f(\xi)\Phi(\xi)e^{i\xi u}d\xi$$

$$=\frac{\sqrt{2\pi}}{\pi}\int_{-\infty}^{\infty}f(\xi)\Phi(\xi)\frac{\sin(x-B-\xi)N}{x-B-\xi}d\xi$$

which is convergent in mean L_q as $N\to\infty$, since $f(\xi)\Phi(\xi)\in L_q$.[2]

Thus we obtained that $G(z)$ is represented by its Poisson integral and on the real axis $G(x)\in L_p$ has its transform in L_q. Hence by another theorem of Hille and Tamarkin[3] the Fourier transform vanishes for negative arguments and

$$G(z)=\int_{0}^{\infty}e^{izx}g(x)dx\,, \tag{3.7}$$

1) Hille and Tamarkin, On a theorem of Paley and Wiener. Annals Math. **34** (1933). A remark on Fourier transforms and functions analytic in a half-plane. Compositio Math., **1** (1934).

2) See for example, Hille and Tamarkin, On the theory of Fourier transforms. Bull. Amer. Math. Soc. **39** (1933). Also see Hille, Offord and Tamarkin, Some observations on the theory of Fourier transforms. ibid. **40** (1934).

3) Hille and Tamarkin, loc. cit. Annals of Math., or Compositio Math. Actually they proved that if $\int_{-\infty}^{\infty}|G(x+iy)|^p dx\leq M=M(p), (y>0)$ and $G(x)$ has the Fourier transform, then (3.7) holds true. Now if (3.6) holds, we can easily verify the uniform integrability (with respect to y) of $|G(x+iy)|^p$. Thus we can apply the theorem. Or it may be noticed directly that to perform the proof, it is sufficient only to assume that the function is represented by its Poisson integral and belongs to L_p on the real axis.

where $g(x)$ is the Fourier transform of $G(x)$ and thus $g(x)=g_\varepsilon(x)\in L_q$. We have

$$H(z)=\frac{1}{\varepsilon}\int_z^{z+\varepsilon}F(w)dw=\int_0^\infty g(x)e^{iz(x-B)}dx$$

$$=\int_{-B}^\infty g(x+B)\,e^{izx}dx\,.$$

And the Fourier transform of $H(x)$ vanishes for $x<-B$.

Similarly

$$H(z)=\int_{-\infty}^B g_1(x-B)\,e^{izx}dx$$

and the Fourier transform also vanish for $x>B$. Hence $H(x)$ is the Fourier transform of a function $h(x)$ which vanishes for $|x|>B$. As already be shown, $H(x)$ has the transform $\sqrt{2\pi}\cdot f(x)\Phi(x)$ which is identical with $h(x)$ by the Fourier transform identity theorem.[1] Since $\sqrt{2\pi}\cdot\Phi(x)$ tends to unity with $\varepsilon\to 0$, $f(x)=0$ for $|x|>B$. Thus

$$H(z)=\frac{1}{\sqrt{2\pi}}\int_{-B}^B f(x)e^{izx}dx\,.$$

Letting $\varepsilon\to 0$, we get

$$F(z)=\frac{1}{\sqrt{2\pi}}\int_{-B}^B f(x)e^{izx}dx\,.$$

Letting $B\to A$, we get the desired result.

4. The proof of Theorem 2 is completely analogous as the one of Theorem 1. Theorem 3 can be easily proved. For that $F(z)=\int_{-A}^A f(x)e^{izx}dx=O\,(e^{A|z|})$ and the analiticity of $F(z)$ are almost evident and if $f(x)\in L_p$ is the Fourier transform of a function $\bar{F}(x)$ in L_q, then the identity theorem shows $\bar{F}(x)=F(x)$ and hence $F(x)\in L_q$. Thus Theorem 3 is proved. Theorem 4 is also immediate.

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1) Berry, Annals of Math., **32** (1931), pp. 227-232.