84. *Application of Mean Ergodic Theorem to the Problems of Markoff's Process.*

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§ **1.** *Two supplements to the Mean Ergodic Theorem.*

Mean Ergodic Theorem. Let $\mathfrak{B}$ *be a (real or complex) Banach space, and denote by* T *a linear operator which maps* $\mathfrak{B}$ *in itself. If*

(1) *there exists a constant* C *such that* $\|T^n\| \leqq C$ *for* $n=1, 2, \ldots,$
and

(2) $\begin{cases} \text{for any } x \in \mathfrak{B} \text{ the sequence } x_n = \frac{1}{n}(T+T^2+\cdots+T^n)x \ (n=1, \\ 2, \ldots) \text{ is weakly compact in } \mathfrak{B}, \end{cases}$

then

(3) $\begin{cases} \text{there exists a linear operator } T_1, \text{ which maps } \mathfrak{B} \text{ in itself, such} \\ \text{that } \lim_{n\to\infty} \frac{1}{n}(T+T^2+\cdots+T^n)x = T_1 x \text{ strongly for any } x \in \mathfrak{B}, \text{ and} \\ TT_1 = T_1T = T_1^2 = T_1. \end{cases}$

T_1 is a projection operator which maps $\mathfrak{B}$ on the proper space $\mathfrak{B}_1$ of T belonging to the proper value 1. Because of (1), (2) is surely satisfied if T is *weakly completely continuous*, viz., if T maps the unit sphere $\|x\| \leqq 1$ of $\mathfrak{B}$ on a point set weakly compact in $\mathfrak{B}$. These results were obtained in our previous notes.[1] We now prove the

Theorem 1. (2) *and hence* (3) *hold good if* T *satisfies* (1) *and if*

(2′) $\begin{cases} \text{there exist an integer } k \text{ and a weakly completely continuous} \\ \text{linear operator } V, \text{ which maps } \mathfrak{B} \text{ in itself, such that} \\ \|T^k - V\| < 1. \end{cases}$

Proof:[2] It is sufficient to prove the case $k=1$. Put $\|T-V\| = \alpha < 1$ and $x_{n,p} = \frac{1}{n}(T+T^2+\cdots+T^p)x$ $(n, p=1, 2, \ldots)$. We have $T^p = V_p + D_p$, where $V_p = T^p - (T-V)^p$ is weakly completely continuous with $V_1 \equiv V$ and $\|D_p\| \leqq \alpha^p$. Hence $x_n = x_{n,p} + T^p x_{n,n-p} = x_{n,p} + V_p x_{n,n-p} + D_p x_{n,n-p}$. Since $\|x_{n,n-p}\| \leqq C \cdot \|x\|$ for $n=1, 2, \ldots$, there exists (for each p) a subsequence $\{n'\}$ of $\{n\}$ such that $\{V_p x_{n',n'-p}\}$ converges weakly to a point $y_p \in \mathfrak{B}$. Consequently we have (since $\lim_{n'\to\infty} |f(x_{n',p})| = 0$)

(4) $$\overline{\lim_{n'\to\infty}} |f(x_{n'}) - f(y_p)| \leqq \lim_{n'\to\infty} |f(x_{n',p})| + \overline{\lim_{n'\to\infty}} |f(D_p x_{n',n'-p})| \leqq \alpha^p \cdot \|f\| \cdot C \cdot \|x\|$$

for any linear functional f on $\mathfrak{B}$.

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1) K. Yosida: Mean Ergodic Theorem in Banach spaces, Proc. **14** (1938), 292.
S. Kakutani: Iteration of linear operations in complex Banach spaces, ibd., 295.

2) Cf. the arguments given by one of us. See the paper of S. Kakutani cited in (1).

Applying the diagonal method, we may assume that (4) holds for any linear functional f on $\mathfrak{B}$ and for $p=1,2,\dots$ (y_p may depend on p). Consider the sequence $\{y_p\}$ $(p=1,2,\dots)$. From (4) we have $|f(y_p)-f(y_q)|\leqq(\alpha^p+\alpha^q)\cdot\|f\|\cdot C\cdot\|x\|$, and, since f is an arbitrary functional on $\mathfrak{B}$, $\|y_p-y_q\|\leqq(\alpha^p+\alpha^q)\cdot C\cdot\|x\|$ for any p and q, which shows that $\{y_p\}$ is a fundamental sequence in $\mathfrak{B}$. Put $y=\lim_{p\to\infty}y_p$. Then it is easy to see that we have $\lim_{n'\to\infty}f(x_{n'})=f(y)$ for any linear functional f on $\mathfrak{B}$. Hence the sequence $\{x_{n'}\}$ converges weakly to a point $y\in\mathfrak{B}$.

Next we shall prove a theorem which constitutes a generalisation of a theorem due to S. Mazur.[1]

Theorem 2. Let T satisfy (1) *and* (2). *Consider the proper value equation*

$$(5)\qquad Tx=x,$$

and its conjugate equation[2]

$$(6)\qquad \bar{T}X=X.$$

Then, if p and q denote the numbers of the linearly independent solutions of (5) *and* (6) *respectively, we must have $p=q$.*

Proof: Put $\mathfrak{B}_1=T_1\mathfrak{B}$. Then $p=$dimension of $\mathfrak{B}_1$. Any linear functional $X(x)$ on $\mathfrak{B}_1$ defines a linear functional $X'(x)$ on $\mathfrak{B}$: $X'(x)=X(T_1x)$. By (3) we have, for any $x\in\mathfrak{B}$, $X'(Tx)-X'(x)=X(T_1Tx)-X(T_1x)=X(T_1x-T_1x)=0$. Hence the linear functional X' satisfies (6), from which follows $q\geqq p$.

Conversely, let X be a linear functional on $\mathfrak{B}$ which satisfies (6). Then we have, for any $x\in\mathfrak{B}$, $X(x)=X(Tx)=X\left(\frac{1}{n}(T+T^2+\cdots+T^n)x\right)$ and hence $X(x)=X(T_1x)$ by (3). Thus X is, essentially, a linear functional on $\mathfrak{B}_1=T_1\mathfrak{B}$, and hence $q\leqq p$.

§ 2. *Applications to the problem of Markoff's process.*

Consider a Markoff's process by which each point x of the closed interval $\Omega=[0,1]$ is transferred to a point $y\in\Omega$ after the elapse of a unit time. Denote by $P(x,E)$ its transition probability; that is, $P(x,E)$ is a probability that a point x comes into a Borel set E of Ω after the elapse of a unit time. We have $0\leqq P(x,E)\leqq 1$ and $P(x,\Omega)=1$. Assume that $P(x,E)$ is measurable in x if E is fixed, and that, for any fixed x, $P(x,E)$ is a totally additive set function defined for all the Borel sets of Ω.

§ 2-1. *Condition of J. L. Doob.*[3]

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1) S. Mazur: Über die Nullstellen linearer Operatoren, Studia Math. **2** (1930), 11–20. The assumption in Theorem 2 is much weaker than that of Mazur's. He assumed that $\mathfrak{B}$ is locally weakly compact.

2) As to the notion of conjugate operators see the paper of S. Mazur cited in (1).

3) J. L. Doob: Stochastic processes with an integral valued parameter, Trans. Amer. Math. Soc. **44** (1938), 87–150.

(7) *There is a measurable function $p(x, y)$ defined for $0 \leqq x, y \leqq 1$ such that $P(x, E) = \int_E p(x, y)\,dy$ for any $x \in \Omega$ and for any Borel set E of Ω; and moreover, $p(x, y)$ satisfies the uniform integrability condition: for any decreasing sequence $\{E_n\}$ of measurable sets with $m(E_n) \to 0$, we have $\int_{E_n} p(x, y)\,dy \to 0$ uniformly in x.*

It will be easily seen that this condition is equivalent to the following one:

(8) *for any positive number $\varepsilon > 0$ there exists a positive number $\delta(\varepsilon) > 0$ such that $m(E) < \delta(\varepsilon)$ implies $P(x, E) < \varepsilon$ for any $x \in \Omega$.*

Theorem 3. Under the condition of Doob, the integral operator

$$f \to Tf = g: \quad g(y) = \int_0^1 f(x)\,p(x, y)\,dx$$

is a linear operator which maps the space (L)[1] in itself. This T is of norm 1 and is weakly completely continuous.

Proof: We have, by Fubini-Tonelli's theorem, $\| g \|_L = \int_0^1 | g(y) |\,dy \leqq \int_0^1\int_0^1 | f(x)\,p(x, y) |\,dx\,dy = \int_0^1 | f(x) | \left(\int_0^1 p(x, y)\,dy\right) dx = \int_0^1 | f(x) |\,dx = \| f \|_L$. Hence $\| T \|_L \leqq 1$[2] By taking $f(x) \equiv 1$ we see that $\| T \|_L = 1$.

As the conjugate space of (L) is the space (M),[3] any linear functional k defined on the image $T(L)$ of (L) is given by

$$\int_0^1 g(y)\,k(y)\,dy = \int_0^1\left(\int_0^1 f(x)\,p(x, y)\,dx\right) k(y)\,dy$$
$$= \int_0^1 f(x)\left(\int_0^1 p(x, y)\,k(y)\,dy\right) dx\,, \quad k(y) \in (M)\,.$$

The subset $(M)'$ of (M) of all the functions of the form: $\int_0^1 p(x, y)\,k(y)\,dy$, $k(y) \in (M)$, $\| k \|_M \leqq 1$, is separable in the topology of (M). This may be proved as follows:

Let S be the unit sphere $\| k \|_M \leqq 1$ of (M). Since $S \subset (M) \subset (L)$ and since (L) is separable, there exists a countable subset $\{k_n(y)\}$ of S which is dense in S in the topology of (L); that is, for any $k(y) \in (M)$ with $\| k \|_M \leqq 1$, there exists a subsequence $\{k_{n'}(y)\}$ of $\{k_n(y)\}$ such that $\lim_{n' \to \infty} \| k - k_{n'} \|_L = \lim_{n' \to \infty} \int_0^1 | k(y) - k_{n'}(y) |\,dy = 0$. Consequently, there exists a further subsequence $\{k_{n''}(y)\}$ of $\{k_{n'}(y)\}$, a decreasing sequence $\{E_{n''}\}$ of measurable sets and a sequence $\{\varepsilon_{n''}\}$ of positive numbers such that $\lim_{n'' \to \infty} m(E_{n''}) = 0$, $\lim_{n'' \to \infty} \varepsilon_{n''} = 0$ and $| k(y) - k_{n''}(y) | \leqq \varepsilon_{n''}$ for any $y \bar{\in} E_{n''}$.

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1) (L) is the linear space of all the measurable functions which are absolutely integrable in $[0, 1]$. For any $f(x) \in (L)$, we define its norm by $\| f \|_L = \int_0^1 | f(x) |\,dx$.

2) $\| T \|_L$ is a norm of T as an operator in (L). Analogous notations will be used for other Banach spaces.

3) (M) is the linear space of all the bounded measurable functions defined in $[0, 1]$. For any $k(x) \in (M)$, we define its norm by $\| k \|_M = \operatorname{ess.\,max.}_{0 \leqq x \leqq 1} | k(x) |$.

Hence $\left|\int_0^1 p(x,y)k(y)\,dy-\int_0^1 p(x,y)k_{n''}(y)\,dy\right|\leqq\int_0^1 p(x,y)\,|k(y)-k_{n''}(y)|\,dy$ $\leqq\int_{E_{n''}}+\int_{\Omega-E_{n''}}\leqq 2\int_{E_{n''}} p(x,y)\,dy+\varepsilon_{n''}\to 0$ uniformly in x. This proves the separability of $(M)'$ in the topology of (M).

Since the space (L) is weakly complete, we see by the diagonal process that T is weakly completely continuous as an operator in (L).

Theorem 4. Under the condition of Doob, there exists a measurable function $p_\infty(x,y)$ *defined for* $0\leqq x,y\leqq 1$ *such that for any* $f(x)\in(L)$ *we have*

$$\lim_{n\to\infty}\int_0^1\left|\int_0^1 f(x)\left(\frac{p(x,y)+p^{(2)}(x,y)+\cdots+p^{(n)}(x,y)}{n}\right)dx-\int_0^1 f(x)p_\infty(x,y)dx\right|dy=0$$

$$\left(p^{(n)}(x,y)=\int_0^1 p^{(n-1)}(x,z)p(z,y)dz,\ n=2,3,\ldots;\ p^{(1)}(x,y)=p(x,y)\right),$$

and

$$\int_0^1 p(x,z)p_\infty(z,y)dz=\int_0^1 p_\infty(x,z)p(z,y)dz=\int_0^1 p_\infty(x,z)p_\infty(z,y)dz=p_\infty(x,y), \tag{9}$$

$$p_\infty(x,y)\geqq 0\,,\quad \int_0^1 p_\infty(x,y)\,dy=1\,. \tag{10}$$

Theorem 5.[1] *Under the condition of Doob, the proper value* λ *of modulus* 1 *of* T *is finite in number and satisfies the binomial equation:* $\lambda^N=1$, *where* N *is a fixed positive integer.*

Proof: The conjugate equation of $f(y)=\lambda\int_0^1 f(x)p(x,y)\,dx$, $f(x)\in(L)$, is given by

$$g(x)=\lambda\int_0^1 p(x,y)g(y)\,dy\,,\quad g(y)\in(M)\,. \tag{11}$$

Hence, by Theorem 2, it is sufficient to show that if (11) admits a solution $g(x)$, $\|g\|_M=1$, we must have $\lambda^n=1$, where n is an integer not greater than some constant determined only by the function $p(x,y)$. This may be done as follows:

For any δ with $0<\delta<1$ we have

$$|g(x)|\leqq\int_{|g(y)|\leqq 1-\delta}p(x,y)(1-\delta)\,dy+\int_{|g(y)|>1-\delta}p(x,y)\,dy$$

$$=\int_0^1 p(x,y)(1-\delta)\,dy+\delta\int_{|g(y)|>1-\delta}p(x,y)\,dy=1-\delta+\delta\int_{|g(y)|>1-\delta}p(x,y)\,dy\,.$$

Since $\|g\|_M=1$, there exists an x_0 with $|g(x_0)|>1-\frac{\delta}{2}$; and for this x_0 we have $1\geqq\int_{|g(y)|>1-\delta}p(x_0,y)\,dy\geqq\frac{1}{2}$. Therefore, by (8), there exists a constant $\gamma>0$ determined from the function $p(x,y)$ only, such that

1) This is a generalisation of a theorem of M. Fréchet. Fréchet assumed that $p^{(n)}(x,y)$ is uniformly bounded. See the paper of Fréchet: Sur l'allure asymptotique des densités itétées dans le problème des probabilités "en chaîne," Bull. de la Soc. math. de France, **62** (1934), 68-83.

$m(E[|g(y)|>1-\delta])>\gamma$ for any $\delta>0$ and for any solution $g(y)$ of (11) with $\|g\|_M=1$.

The rest of the proof may be carried out as in the paper of Fréchet's.

Theorem 6. Under the condition of Doob, the proper value 1 *of* T *is of finite multiplicity.*

Proof: First we notice that there is a constant $\gamma>0$ such that $\int_E p(x,y)\,dy=1$ implies $m(E)\geqq\gamma$ for any x and for any measurable set $E\subset\Omega$.

Let now $f(y)=\int_0^1 f(x)\,p(x,y)\,dx$. Then $|f(y)|\leqq\int_0^1|f(x)|\,p(x,y)\,dx$, and hence by integrating with respect to y and applying Fubini-Tonelli's theorem, we see that $|f(y)|=\int_0^1|f(x)|\,p(x,y)dx$ almost everywhere. Therefore, following the same arguments as were given by N. Kryloff and N. Bogolioùboff,[1] we see that, if the multiplicity of the proper value 1 is greater than $>\frac{1}{\gamma}$, there exists a system of $n\left(>\frac{1}{\gamma}\right)$ real non-negative measurable functions satisfying

$$\int_0^1 p_i(y)\,dy=1\,,\quad p_i(y)\cdot p_j(y)\equiv 0 \ \text{ for } i\neq j\,,$$

and
$$p_i(y)=\int_0^1 p_i(x)\,p(x,y)\,dx \ \text{ almost everywhere.}$$

Let E_i be the set of y at which $p_i(y)>0$, then E_i are mutually disjoint. We have $1=\int_{E_i}p_i(y)\,dy=\int_{E_i}\left(\int_{E_i}p_i(x)\,p(x,y)\,dx\right)dy=\int_{E_i}p_i(x)\left(\int_{E_i}p(x,y)\,dy\right)dx$ by Fubini-Tonelli's theorem. Hence $\int_{E_i}p(x,y)\,dy=1$ almost everywhere in E_i, and thus $m(E_i)\geqq\gamma$ for $i=1,2,\dots,n$, which is a contradiction since $n>\frac{1}{\gamma}$. Consequently the multiplicity of the proper value 1 is not greater than $\frac{1}{\gamma}$.

§ 2-2. *Condition of W. Doeblin.*[2]

(12) $\left\{\right.$ *There exist two positive numbers* $b,\eta>0$ *such that* $m(E)<\eta$ *implies* $P(x,E)<1-b$ *for any* x *and for any measurable set* $E\subset\Omega$.

Clearly the condition of Doeblin is much more general than that of Doob.

Theorem 7. Under the condition of Doeblin, the integral operator

$$\varphi\to T\varphi=\psi:\quad \psi(E)=\int_0^1\varphi(de_x)\,P(x,E)$$

1) N. Kryloff and N. Bogolioùboff: Sur les propriétés ergodiques de l'equation de Smoluchovski, Bull. de la Soc. math. de France, **64** (1936), 49–56.

2) W. Doeblin: Sur les propriétés asymptotiques de mouvements régis par certains types de chaines simples, Bull. math. de la Soc. Roumaine des Sciences, **39** (1937), (2), 3–61.

is a linear operator which maps the space $(\mathfrak{M})$[1] *in itself. This T is of norm* 1 *and there exists a weakly completely continuous linear operator V, which maps* $(\mathfrak{M})$ *in itself, such that* $\| T-V \|_{\mathfrak{M}} < 1$.

The same theorem may be stated for the space (BV).[2] For this purpose, denote by $I(y_0)$ the closed interval $0 \leqq y \leqq y_0$ and consider the function $F(x, y) \equiv P\big(x, I(y)\big)$. $F(x, y)$ is a measurable function defined for $0 \leqq x, y \leqq 1$, and is monotone in y if x is fixed.

Theorem 7′. Under the condition of Doeblin, the integral operator

$$\varphi \rightarrow T\varphi = \psi: \quad \psi(y) = \int_0^1 \varphi(dx) F(x, y)$$

is a linear operator which maps the space (BV) *in itself. This T is of norm* 1 *and there exists a weakly completely continuous linear operator V, which maps* (BV) *in itself, such that* $\| T-V \|_{BV} < 1$.

Proof: Since $F(x, y)$ is monotone in y if x is fixed, $p(x, y) = \dfrac{\partial F}{\partial y}$ exists almost everywhere (for each x). $p(x, y)$ is measurable in $0 \leqq x, y \leqq 1$. Put $q(x, y) = p(x, y)$ if $p(x, y) \leqq \dfrac{1}{\eta}$ and $q(x, y) = 0$ if $p(x, y) > \dfrac{1}{\eta}$. Then $G(x, y) = \int_0^y q(x, t)\,dt$ and $H(x, y) = F(x, y) - G(x, y)$ are also measurable in $0 \leqq x, y \leqq 1$, and monotone in y if x is fixed. Now, consider the linear operators V and W which correspond to $G(x, y)$ and $H(x, y)$ respectively. Clearly $T = V + W$. We shall show that V is weakly completely continuous as an operator in (BV) and that $\| W \|_{BV} \leqq 1 - b < 1$.

In order to prove that V is weakly completely continuous, let $\{\varphi_n(x)\}$ be a sequence of functions of bounded variation with $\| \varphi_n \|_{BV} \leqq 1$, $n = 1, 2, \ldots$. We have to choose a subsequence $\{\varphi_{n'}(x)\}$ of $\{\varphi_n(x)\}$ and a function $\varphi_0(x) \in (BV)$ such that $V\varphi_{n'}$ converges weakly to $\varphi_0(x)$; that is, $f(\varphi_{n'})$ converges to $f(\varphi_0)$ for any linear functional f on (BV). It is disappointing that the general form of linear functionals on (BV) is not yet known, but we can evade this difficulty. Since $V\varphi$ is absolutely continuous for any $\varphi(x) \in (BV)$, and since the subspace (A) of (BV) of all the absolutely continuous functions of (BV) is isometric to (L), f may be considered as a functional on (L); and consequently, by a well-known result, f is represented by a function $k(x) \in (M)$. Moreover, since $V\varphi$ is absolutely continuous with uniformly bounded density $\left(\leqq \dfrac{1}{\eta}\right)$ for any $\varphi(x) \in (BV)$ with $\| \varphi \|_{BV} \leqq 1$, the range of V corresponding to a sphere $\| \varphi \|_{BV} \leqq 1$ of (BV) is even isometric to a uniformly bounded $\big($in the topology of $(M)\big)$ part of (M), which is a linear subspace of (L).

Thus our problem is transformed into the following one: Given

1) $(\mathfrak{M})$ is the linear space of all the totally additive set functions defined for all the Borel sets of $\Omega = [0, 1]$. For any $\varphi(E) \in (\mathfrak{M})$, we define its norm by $\| \varphi \|_{\mathfrak{M}}$ = total variation of $\varphi(E) \equiv$ l. u. b. $\varphi(E)$ − g. l. b. $\varphi(E)$.

2) (BV) is the linear space of all the functions of bounded variation defined in $0 \leqq x \leqq 1$. For any $\varphi(x) \in (BV)$, we define its norm by $\| \varphi \|_{BV} = | \varphi(0) |$ + total variation of $\varphi(x)$ in $0 \leqq x \leqq 1$.

a sequence $\{g_n(x)\}$ of uniformly bounded measurable functions, $g_n(x) \in (M)$, we have to choose a subsequence $\{g_{n'}(x)\}$ of $\{g_n(x)\}$ and a function $g_0(x) \in (L)$,[1] such that we have $\lim_{n' \to \infty} \int_0^1 g_{n'}(x)\, k(x)\, dx = \int_0^1 g_0(x)\, k(x)\, dx$ for any function $k(x) \in (M)$.

This problem may be solved as follows: Since $(M) \subset (L)$ and since (L) is separable, there is a countable subset $\{k_m(x)\}$ of (M) which is dense in (M) in the topology of (L). Applying the diagonal method, we can choose a subsequence $\{g_{n'}(x)\}$ of $\{g_n(x)\}$ such that $\lim_{n' \to \infty} \int_0^1 g_{n'}(x)\, k_m(x)\, dx$ exists for $m = 1, 2, \dots$. Since $\{g_{n'}(x)\}$ is uniformly bounded, $\lim_{n' \to \infty} \int_0^1 g_{n'}(x)\, k(x)\, dx$ exists for any $k(x) \in (M)$. The existence of a limiting function $g_0(x) \in (L)$ is now a direct consequence of the facts that $(M) \subset (L)$ and that (L) is weakly complete.

In order to prove that $\|W\|_{BV} \leqq 1 - b < 1$, it is sufficient to show that $H(x, 1) \leqq 1 - b$ for any x. This may be easily seen from the condition of Doeblin, if we observe that, for any x, the set of y, where $H(x, y)$ actually increases, is of measure $< \eta$ by the construction of $H(x, y)$ $\big($and $q(x, y)\big)$.

Remark. Theorems 4 and 6 are also true for the case when the condition of Doeblin is satisfied. This may be easily seen as in the preceding.

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1) In general, it is impossible to take $g_0(x)$ in (M).