91. *Operator-theoretical Treatment of the Markoff's Process.*

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§ 1. *Introduction.* Let $P(x, E)$ denote the transition probability that the point x of the interval $(0, 1)$ is transferred, by a simple Markoff's process, into the Borel set E on the interval $(0, 1)$ after the elapse of a unit time. We assume that $P(x, E)$ is completely additive for Borel sets E if x is fixed and that $P(x, E)$ is Borel measurable in x if E is fixed. Then the transition probability after the elapse of n units of time is given by $P^{(n)}(x, E)=\int_0^1 P^{(n-1)}(x, dy)P(y, E)$, $\left(P^{(1)}(x, E)=P(x, E)\right)$.

Under certain general condition given below, W. Doeblin[1] investigated the asymptotic behaviour of $P^{(n)}(x, E)$ for large n. His method of proof is based upon set-theoretical considerations. It may be termed as a direct method. In the present note I intend to give an operator-theoretical treatment of the problem, by virtue of the results of the preceding notes.[2] Our method of proof will make clear the spectral properties of the Markoff's process in question, and the results obtained is somewhat more precise than that of Doeblin. The author is indebted to S. Kakutani in the proof of the lemma 1 and 5 below. I want to express my hearty thanks to him.

§ 2. *Preliminary lemmas.* By definition we have

(1) $P^{(n)}(x, E)\geqq 0$ and $P^{(n)}(x, \Omega)\equiv 1$, where $\Omega=$ the interval $(0, 1)$.

We make on $P(x, E)$ the following assumptions due to Doeblin:

$$(2)\quad \begin{cases} \text{there exist an integer } s \text{ and positive } b, \eta\ (<1) \text{ such that,} \\ \text{if } \operatorname{mes}(E)<\eta, \quad P^{(s)}(x, E)<1-b \quad \text{uniformly in } x. \end{cases}$$

Then it is easy to see that

$$(2)' \qquad \text{if } \operatorname{mes}(E)<\eta, \quad P^{(t)}(x, E)<1-b \quad \text{uniformly in } x \text{ and } t\geqq s.$$

We may decompose $P^{(s)}(x, E)$ as follows[3]:

$$(3)\quad \begin{cases} P^{(s)}(x, E)=\int_E q(x, y)dy+R(x, E), \\ 0\leqq q(x, y)\leqq \dfrac{1}{\eta}, \quad 0\leqq R(x, E)<1-b. \end{cases}$$

1) W. Doeblin: Sur les propriétés asymptotiques de mouvements régis par certains types de chaines simples, Bull. math. de la Soc. Roumaine des Sciences, **39** (1937), (2), 3–61.

2) K. Yosida: Abstract integral equations and the homogeneous stochastic process, Proc. **14** (1938), 286. K. Yosida and S. Kakutani: Application of mean ergodic theorem to the problem of Markoff's process, ibid. 333. K. Yosida, Y. Mimura and S. Kakutani: Integral operator with bounded kernels, ibid. 359. These notes will respectively be referred to as [I], [II] and [III].

3) [II], the proof of Theorem 7′.

Consider the integral operator P which transforms the Banach space $(\mathfrak{M})$ in $(\mathfrak{M})$[1]:

$$P\cdot f=g, \qquad g(E)=\int_0^1 P(x, E)f(dx).$$

By (1) and (3) we have

$$\begin{cases} P^s=Q+R, \quad \|P\|=1, \quad \|Q\|\leqq 1, \quad \|R\|\leqq 1-b, \quad \text{where} \\ Q\cdot f=\int_E dy\int_0^1 q(x, y)f(dx), \quad R\cdot f=\int_0^1 R(x, E)f(dx). \end{cases}$$

Lemma 1. There exist an integer n and a completely continuous operator V such that $\|P^n-V\|<1$.

Proof. We have $P^{sk}=Q^k+Q^{k-1}R+Q^{k-2}RQ+\cdots+RQR^{k-2}+QR^{k-1}+R^k$. The term which contains Q at least two times as factor is *completely continuous*. Consider, for example, $RQRQ^{k-3}$. QR and Q^{k-3} are both integral operators with bounded kernels. Hence[2] QRQ^{k-3} is completely continuous in $(\mathfrak{M})$, and thus $RQRQ^{k-3}$ is also completely continuous in $(\mathfrak{M})$. The number of terms that contains Q at most once as factor is $k+1$, each of norm $\leq(1-b)^{k-1}$. As $\lim_{k\to\infty}(k+1)(1-b)^{k-1}=0$ we have the lemma.

Lemma 2. Let $\|P^n-V\|<1$ as in lemma 1. Then, if $k\geq n$, the proper value λ with modulus 1 of P^k satisfies $\lambda^{m_k}=1$, where the integer m_k is bounded uniformly for k.

Proof. We have $\|P^k-P^{k-n}V\|\leq\|P^{k-n}\|\,\|P^n-V\|\leq\|P^n-V\|<1$. $P^{k-n}V$ is completely continuous with V. Let $\lambda(|\lambda|=1)$ be a proper value of P^k. Then[3] there exists a projection operator $P_{k,\lambda}(x, E)$ which maps $(\mathfrak{M})$ on the proper space belonging to the proper value λ of P^k:

$$\begin{cases} P_{k,\lambda}(x, E)=\lim_{t\to\infty}\frac{1}{t}\sum_{i=1}^{t}P^{(ki)}(x, E) \qquad \text{(uniform limit)}, \\ \lambda P_{k,\lambda}(x, E)=\int_0^1 P_{k,\lambda}(y, E)P^{(k)}(x, dy)=\int_0^1 P^{(k)}(x, dy)P_{k,\lambda}(y, E), \\ P_{k,\lambda}(x, E)=\int_0^1 P_{k,\lambda}(x, dy)P_{k,\lambda}(y, E). \end{cases}$$

Hence, for any $g(y)$ of the Banach space $(\mathfrak{M})$, the element in $h(x)=\int_0^1 P_{k,\lambda}(x, dy)g(y)$ of $(\mathfrak{M})$ satisfies $h(x)=\lambda\int_0^1 P^{(k)}(x, dz)h(dz)$. Therefore, by (1)′, we obtain the lemma by applying M. Fréchet's arguments.[4]

Lemma 3. Let $\|P^n-V\|<1$ as in lemma 1. Then, for any $k\geq n$, the multiplicity of the proper value 1 of P^k is $<\frac{1}{\eta}$.

Proof. Let $f(E)=\int_0^1 P^{(k)}(x, E)f(dx)$. We have, by (1), $g(E)=$

1) $(\mathfrak{M})$ is the Banach space of all the totally additive set functions defined for all the Borel sets of $\Omega=(0,1)$. For any $f(E)\in(\mathfrak{M})$, we define its norm by $\|f\|_{\mathfrak{M}}=$total variation of f on $(0,1)$.

2) [III], Theorem 2.

3) [I], Theorem. Cf. also Theorem 4 in S. Kakutani: Iteration of linear operations in complex Banach spaces, Proc. **14** (1938), 295.

4) [II], Theorem 5.

$\int_0^1 P^{(k)}(x, E)g(dx)$, where $g(E)$ = the total variation of f on E. Hence, by $(1)'$, we obtain the lemma by applying Kryloff-Bogoliouboff's arguments.[1]

Kryloff-Bogoliouboff's arguments for the proof of the above lemma also proves the following lemma simultaneously.

Lemma 4. Let $\| P^n - V \| < 1$ as in lemma 1. Then, for any $k \geq n$, there exists $f_{1_k}(E), f_{2_k}(E), \dots, f_{l_k}(E) \left(l_k < \frac{1}{\eta}\right)$ with the properties:

$$P^k \cdot f_{i_k} = f_{i_k}, \; f_{i_k}(E) \geqq 0, \; f_{i_k}(\Omega) = 1, \; f_{i_k}(E) f_{j_k}(E) \equiv 0 \quad \text{for} \quad i \neq j,$$

such that any $f(E)$ satisfying $P^k \cdot f = f$, $f(E) \geqq 0$, $f(\Omega) = 1$ is uniquely expressed as a linear combination $f(E) = \sum_{i_k=1}^{l_k} c_{i_k} f_{i_k}(E)$, $c_{i_k} \geqq 0$, $\sum_{i_k=1}^{l_k} c_{i_k} = 1$.

Lemma 5. Let square matrix $C = \| c_{ij} \|$ $(i, j = 1, 2, \dots, l_n)$ satisfy the conditions: $c_{ij} \geqq 0$, $\sum_{j=1}^{l_n} c_{ij} = 1$ $(i = 1, 2, \dots, l_n)$, C^m = unit matrix for a certain m. Then C represents a permutation of l_n indices $1, 2, \dots, l_n$.

Proof. The matrix C^{m-1} also satisfies the same condition as C. Thus we obtain the lemma by performing the multiplication $C^{m-1} \cdot C =$ the unit matrix.

§ 3. *Asymptotic behaviour of the process P.* By lemma 1, 2 and 3 we see that there exist an integer n and *a completely continuous* operator V such that i) $\| P^n - V \| < 1$, ii) P^n admits of no proper values with modulus 1 other than 1. Hence we have[2]

$$(4) \quad \begin{cases} P^n = P_1 + S, \quad P^n P_1 = P_1 P^n = P_1^2 = P_1, \quad P_1 S = S P_1 = 0, \\ \| S^k \| \leq \dfrac{a}{(1+\varepsilon)^k} \qquad (k = 1, 2, \dots) \text{ with positive } a \text{ and } k. \end{cases}$$

Here the integral operator P_1 is defined by the kernel

$$(5) \qquad P_1(x, E) = \lim_{t \to \infty} \frac{1}{t} \sum_{i=1}^{t} P^{(ni)}(x, E) \qquad \text{(uniform limit)}.$$

Surely we have $P_1(x, E) \geqq 0, P_1(x, \Omega) \equiv 1$ and thus, by lemma 4,

$$(6) \quad \begin{cases} P_1(x, E) = \sum_{i=1}^{l_n} c_i(x) f_i(E) \qquad \left(l_n < \dfrac{1}{\eta}\right), \\ P_1 \cdot f_i = f_i, \quad f_i(E) \geqq 0, \quad f_i(\Omega) = 1, \quad f_i(E) f_j(E) \equiv 0 \quad \text{for} \quad i \neq j, \\ c_i(x) \geqq 0, \quad \sum_{i=1}^{l_n} c_i(x) \equiv 1. \end{cases}$$

We put $P \cdot f_i = g_i$. Then, by $P^n \cdot P = P \cdot P^n$, we have $P^n \cdot g_i = g_i$. It is easy to see that $g_i(E) \geqq 0$, $g_i(\Omega) = 1$. Hence by lemma 4

$$P \cdot f_i = \sum_{j=1}^{l_n} c_{ij} f_j, \quad c_{ij} \geqq 0, \quad \sum_{j=1}^{l_n} c_{ij} = 1 \quad (i = 1, 2, \dots, l_n).$$

The matrix $C = \| c_{ij} \|$ satisfies C^n = the unit matrix by $P^n \cdot f_i = f_i$. Hence, by lemma 5, the l_n indices $1, 2, \dots, l_n$ is divided into p classes $(p \leqq l_n)$, each class being permutated *cyclically* in itself by C. Let K_a

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1) [II], Theorem 6.

2) [I], Theorem. Cf. also Theorem 4 in S. Kakutani: loc. cit.

be the class which consists of the indices $1_a, 2_a, \dots, d_a$: $\sum_{d=1}^{p} d_a = l_n$. As C^n = the identical permutation we see that each d_a is a divisor of n.

$f_i(E)$ is completely additive and non-negative. Hence, by Radon-Nikodym's theorem, $f_i(E) = \int_E p_i(x)dx + f_i(E \cdot N_i)$ with $p_i(x) \geqq 0$ and $\text{mes}(N_i) = 0$. Let $\bar{E}_i$ be the set at each point of which we have $p_i(x) > 0$. We put $E_i = \bar{E}_i + N_i$, $G_a = \sum_{i \in K_a} E_i$. By (6) E_i $(i = 1, 2, \dots, l_n)$ are *mutually disjoint*, and hence G_a $(a = 1, 2, \dots, p)$ are also mutually disjoint. G_a $(a = 1, 2, \dots, p)$ are called the *final sets* of the Markoff's process P.

We now prove the following properties of the final sets.

i). $\underset{x \in \Omega}{\text{l. u. b.}}\, P^{(t)}(x, \Omega - \sum_{d=1}^{p} G_a) \leq \dfrac{a'}{(1+\varepsilon')^t}$ $(t = 1, 2, \dots)$ with positive a', ε'.

ii). $P(x, G_a) = 1$ for almost all $x \in G_a$.

iii). There exists $E'_i \subset E_i$ with $\text{mes}(E_i - E'_i) = 0$ such that, if $i \in K_a$,

$$\underset{x \in E'_i,\, E \subset E_i}{\text{l. u. b.}} | P^{(td_a)}(x, E) - f_i(E) | \leq \frac{a''}{(1+\varepsilon'')^t} \qquad (t = 1, 2, \dots),$$

where a'' and ε'' denote positive constants.

iv). Let $G'_a = \sum_{i \in K_a} E'_i$, then $G'_a \subset G_a$ and $\text{mes}(G_a - G'_a) = 0$. For any $E \subset G_a$, $\text{mes}(E) > 0$ and for any $x \in G'_a$, there exists a positive integer $m = m(x, E)$ such that $P^{(m)}(x, E) > 0$.

v). By a suitable numerotation of the indices $1_a, 2_a, \dots, d_a$, we have $P(x, E_{(i+1)_a}) = 1$ for almost all $x \in E_{i_a}$ $\big(i = 1, 2, \dots, d$, $(d+1)_a = 1_a$, $a = 1, 2, \dots, p\big)$.

vi). By *ii*) $P(x, E)$ defines a Markoff's process P_a in G_a $(a = 1, 2, \dots, p)$. P_a admits of no proper values with modulus 1 other than 1, if and only if $d_a = 1$.

Proof of i). By (4) and (6) we have $P^{(ni)}(x, \Omega - \sum_{a=1}^{p} G_a) = P^{(ni)}(x, \Omega - \sum_{i=1}^{l_n} E_i) = S^{(i)}(x, \Omega - \sum_{i=1}^{l_n} E_i)$. Hence, by (4) and $\|P\| = 1$, we obtain *i*).

Proof of ii). By the definition of the class K_a we have $h_a(E) = \int_0^1 P(x, E) h_a(dx)$, where $h_a(E) = f_{1_a}(E) + f_{2_a}(E) + \cdots + f_{d_a}(E)$. By putting $E = G_a = \sum_{i \in K_a} E_i$ we obtain $1 = \int_0^1 P(x, G_a) h_a(dx)$. We have $1 \geq h_a(E) \geqq 0$, $h_a(G_a) = 1$ and $1 \geqq P(x. G_a) \geqq 0$. Hence we must have $P(x, G_a) = 1$ for almost all $x \in G_a$.

Proof of iii). We obtain, by (5) and (6), $f_i(E) = \int_0^1 P_1(x, E) f_i(dx)$. Hence, by (6) $f_i(E) = \int_0^1 c_i(x) f_i(E) f_i(dx)$ if $E \subset E_i$. Hence, by putting $E = E_i$, $1 = \int_0^1 c_i(x) f_i(dx)$. Thus, by $1 \geqq c_i(x) \geqq 0$, $1 \geqq f_i(E) \geqq 0$ and $f_i(E_i) = 1$ we obtain $c_i(x) = 1$ for almost all $x \in E_i$. Therefore there exists $E'_i \subset E_i$ with $\text{mes}(E_i - E'_i) = 0$ such that $P_1(x, E) = c_i(x) f_i(E) = f_i(E)$ for $x \in E'_1$ if $E \subset E_i$. Then by (4) we have $P^{(nk)}(x, E) = f_i(E) + S^{(k)}(x, E)$ for $x \in E'_i$

and $E \subset E_i$. As d_a is a divisor of n we put $n = d_a s_a$. Then any integer of the form td_a can be expressed as $td_a = m_t n + r_t d_a$, $s_a > r_t \geqq 0$. Thus $P^{(td_a)} = (P^{d_a})^{r_t} (P^n)^{m_t}$. As $P^{d_a} \cdot f_i = f_i$ if $i \in K_a$, we obtain iv) by (6) and $\| P \| = 1$.

Proof of iv). From the above arguments we have $P_1(x, E) = \sum_{i \in K_a} f_i(E)$ for $x \in G'_a$, $E \subset G_a$. This proves iv) by (6).

Proof of v). By the definition of the class K_a, we have $f_{(i+1)_a}(E) = \int_0^1 P(x, E) f_{i_a}(dx)$, by a suitable numerotation of the indices $1_a, 2_a, \dots, d_a$ $\left((d+1)_a = 1_a\right)$. Hence we obtain $1 = \int_0^1 P(x, E_{(i+1)_a}) f_{i_a}(dx)$. As $1 \geqq f_{i_a}(E) \geqq 0$, $f_{i_a}(E_{i_a}) = 1$ and $1 \geqq P(x, E_{(i+1)_a}) \geqq 0$ we must have v).

Proof of vi). Let $d_a = 1$. Then, by iii), the operator P_a does not admit of proper values with modulus 1 other than 1. Let $d_a > 1$. We put, as in the poof of v), $P \cdot f_{i_a} = f_{(i+1)_a}$ $\left(i = 1, 2, \dots, d, (d+1)_a = 1_a\right)$. We have $P \cdot (f_{1_a} + \lambda f_{2_a} + \lambda^2 f_{3_a} + \cdots + \lambda^{d_a - 1} f_{d_a}) = \lambda (f_{1_a} + \lambda f_{2_a} + \lambda^2 f_{3_a} + \cdots + \lambda^{d_a - 1} f_{d_a})$ for any λ with $\lambda^{d_a} = 1$. As $f_{1_a}, f_{2_a}, \dots, f_{d_a}$ are linearly independent we see that there exists at least one proper value $\lambda (\lambda \neq 1)$ with $\lambda^{d_a} = 1$. (E. Schmidt's arguments.)