PAPERS COMMUNICATED

28. *A Problem of Diophantine Approximations in the Old Japanese Mathematics.*

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In studying the history of the old Japanese mathematics, so-called Wazan, I have found that a manuscript with the title *Ruiyaku-zitu* (累約術), or the method of successive divisions, written by *Katahiro Takebe* (建部賢弘, 1664–1739), revised in 1728 by his pupil, *Genkei Nakane* (中根元圭, 1662–1733), contains problems of Diophantine approximations. This manuscript has been mentioned by many mathematicians in our country, but it seems that the importance of its content was not sufficiently perceived by them.

This manuscript consists of the following three problems. The first treats of the integral solutions of $|ax-by|<1$, while the second and the third those of $|ax-by+c|<1$ and $|ax-by-c|<1$ respectively, where a, b, c are given positive real numbers.

The author of this manuscript solved the first problem by expanding b/a into simple continued fractions, quite similar to the modern theory of continued fractions.

For the second and the third problems *Takebe* developed an algorithm very similar to the *Jacobi algorithm* and gave the concrete form for the solutions, which is very remarkable.

I will translate freely the second problem in the following lines.

Problem. Let $c=5513.9106$, the initial additive number (原益數), be added repeatedly by the successive additive number (累益數) $a=954.5338$ and subtracted repeatedly by the successive subtractive number (累損數) $b=6034.4574$. What are the integral values of x, y, which are called the additive multiplier (益段數) and the subtractive multiplier (損段數), such that $ax-by+c$ lies between two given limits (許限) $-\delta$ and $+\delta$? [Here it is assumed $b>c$, $\delta=1$].[1]

The solutions x, y of $0<ax-by+c<1$ are called the strong additive and subtractive multipliers (强益段, 强損段), while the solutions x, y of $-1<ax-by+c<0$ the weak additive and subtractive multipliers (弱益段, 弱損段).

Answer: the strong additive multiplier 15034,
the strong subtractive multiplier 2379,
the weak additive multiplier 854,
the weak subtractive multiplier 136.

Solutions: Since there is the initial additive number c, we solve this problem by two processes.

The first process runs similarly as the first problem. Divide b by a and let the quotient (商) be a_1, the remainder (不盡) be r_1. Next

1) [ ] is the remark of the author of this paper.

divide a by r_1 and let the quotient be a_2, the remainder r_2 and so on.

$$\left[\text{Thus}\quad \frac{b}{a}=a_1+\frac{r_1}{a},\quad \frac{a}{r_1}=a_2+\frac{r_2}{r_1},\quad \frac{r_1}{r_2}=a_3+\frac{r_3}{r_2},\ \cdots\cdots,\right.$$

$$\left.\text{that is}\quad \frac{b}{a}=a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots\cdots\right].$$

Let $p_1=a_1,\quad p_2=1+a_1a_2,\ \cdots,\quad p_n=a_np_{n-1}+p_{n-2},$

$q_1=1,\quad q_2=a_2,\ \cdots\cdots\cdots,\quad q_n=a_nq_{n-1}+q_{n-2}.$

$p_1, p_2, p_3, \cdots$ are called the additive multipliers, while
$q_1, q_2, q_3, \cdots$ the subtractive multipliers.

The second process: $b-c$ is called the initial subtractive number (原損數). Divide $b-c$ by a and let the quotient be b_1', the remainder s_1'; put $b_1=b_1'+1$, $s_1=a-s_1'$. Next divide s_1 by r_1, and let the quotient be b_2', the remainder s_2'; put $b_2=b_2'+1$, $s_2=r_1-s_2'$. And so on.

$$\left[\text{Thus}\quad \frac{b-c}{a}=b_1'+\frac{s_1'}{a}=b_1-\frac{s_1}{a},\quad \frac{s_1}{r_1}=b_2'+\frac{s_2'}{r_1}=b_2-\frac{s_2}{r_1},\right.$$

$$\left.\cdots\cdots\cdots\quad \frac{s_{n-1}}{r_{n-1}}=b_n'+\frac{s_n'}{r_{n-1}}=b_n-\frac{s_n}{r_{n-1}}\right].$$

$b_1', b_2', b_3', \cdots$ are called the abundant quotient (盈商), while
$b_1, b_2, b_3, \cdots$ are called the deficient quotient (朒商),
$s_1', s_2, s_3', s_4, s_5', \cdots$ are called the weak remainder (弱不盡),
$s_1, s_2', s_3, s_4', \cdots$ the strong remainder (强不盡).

Again let $u_1'=b_1',\quad u_2'=p_1b_2'+u_1,\ \cdots,\quad u_n'=p_{n-1}b_n'+u_{n-1},$

$u_1=b_1,\quad u_2=p_1b_2+u_1,\ \cdots,\quad u_n=p_{n-1}b_n+u_{n-1},$

which are called the abundant and the deficient additive multipliers (盈益叚). Further put

$v_1'=1,\quad v_2'=q_1b_2'+v_1,\ \cdots,\quad v_n'=q_{n-1}b_n'+v_{n-1},$

$v_1=1,\quad v_2=q_1b_2+v_1,\ \cdots,\quad v_n=q_{n-1}b_n+v_{n-1},$

which are called the abundant and the deficient subtractive multipliers (損朒叚).

The required solutions are to be found among (u_n, v_n) and (u_n', v_n').

$a=954.5338\qquad b=6034.4574\qquad b-c=520.5458$[1)]

a_i	r_i	p_i	q_i	b_i'	b_i	s_i'	s_i	u_i'	v_i'	u_i	v_i
6	307.2546	6	1	0	1	520.5458	433.9870	0	1	1	1
3	32.7700	19	3	1	2	126.7324	180.5222	7	2	13	3
9	12.3246	177	28	5	6	16.6722	16.0978	108	18	127	21
2	8.1208	373	59	1	2	3.7732	8.5514	304	49	481	77
1	4.2038	550	87	1	2	0.4306	7.6902	854	136	1227	195
1	3.9170	923	146	1	2	3.4864	0.7174	1777	282	2327	369
	0.2868	1473	233	0	1	0.7174	3.1996	2327	369	3250	515
				8[2)]	9	0.9052		15034	2379		

1) In the manuscript the value $c=5513.9116$ is erroneously written 5513.9106.

2) To obtain the values x, y as small as possible, it is here given 8 as quotient instead of 11. See the remark in the end of this paper.

In the third problem $|ax-by-c|<1$, c is taken as the initial subtractive number instead of $b-c$, the remaining part is unchanged.

There is no proof in the manuscript, but perhaps *Takebe* has obtained these results by inductions. I will verify it in the following lines.

For the sake of simplicity I will change somewhat the notation in the following form.

Let α, β be any positive real numbers, and we consider the linear form $x-\alpha\, y-\beta$, which is the case of the third problem. For the second problem it is considered as $ax-by+c=ax-by'-c'$, $y'=y-1$, $c'=b-c$.

$$\text{Let}\quad \alpha=a_1+\omega_1\,,\quad \frac{1}{\omega_1}=a_2+\omega_2\,,\ \cdots,\quad \frac{1}{\omega_{n-1}}=a_n+\omega_n\,,\quad (0<\omega_i<1)\,,$$

$$\beta=b_1-\Omega_1\,,\quad \frac{\Omega_1}{\omega_1}=b_2-\Omega_2\,,\ \cdots,\quad \frac{\Omega_{n-1}}{\omega_{n-1}}=b_n-\Omega_n\,,\quad (0<\Omega_i<1)\,.$$

If we put $b_i'=b_i-1$, $\Omega_i'=1-\Omega_i$, then

$$\beta=b_1'+\Omega_1'\,,\quad \frac{\Omega_1}{\omega_1}=b_2'+\Omega_2'\,,\ \ldots\ldots.$$

$$\text{Let}\qquad \frac{p_n}{q_n}=a_1+\frac{1}{a_2}\,\frac{}{+}\,\frac{1}{a_3}\,\frac{}{+}\cdots\frac{1}{+\,a_n}\,,$$

then it is easily seen that

$$\alpha=\frac{\omega_n p_{n-1}+p_n}{\omega_n q_{n-1}+q_n}\,,$$

$$\beta=\frac{A_n+B_n\omega_n+(-1)^n\Omega_n}{\omega_n q_{n-1}+q_n}\,,$$

where

$$p_{n+1}=a_{n+1}p_n+p_{n-1}\,,\quad q_{n+1}=a_{n+1}q_n+q_{n-1}\,,$$

$$A_{n+1}=a_{n+1}A_n+B_n+(-1)^n b_{n+1}\,,\quad A_1=b_1\,,$$

$$B_{n+1}=A_n\,,\qquad B_1=0\,.$$

Putting these values of α, β in $x-\alpha\, y-\beta$, we have

$$x-\alpha\, y-\beta=(\omega_n q_{n-1}+q_n)^{-1}\{x(\omega_n q_{n-1}+q_n)-y(\omega_n p_{n-1}+p_n)$$
$$-(A_n+B_n\omega_n+(-1)^n\Omega_n)\}\,.$$

If we define u_n, v_n by

$$u_n q_n-v_n p_n=A_n\,,\quad u_n q_{n-1}-v_n p_{n-1}=B_n\,,$$

that is

$$u_n=(-1)^{n+1}(p_{n-1}A_n-p_nB_n)\,,$$

$$v_n=(-1)^{n+1}(q_{n-1}A_n-q_nB_n)\,,$$

we have

$$u_n-\alpha\, v_n-\beta=(-1)^{n+1}\Omega_n(\omega_n q_{n-1}+q_n)^{-1}\,.$$

From the recurring formula for A_n, B_n, we obtain

$$u_{n+1}=p_nb_{n+1}+u_n\,,\quad v_{n+1}=q_nb_{n+1}+v_n\,,\quad u_1=b_1\,,\quad v_1=1\,.$$

If we put $$u'_{n+1}=p_n b'_{n+1}+u_n\,,\quad v'_{n+1}=q_n q'_{n+1}+v_n\,,$$
$$=u_{n+1}-p_n\,,\qquad =v_{n+1}-q_n\,,$$
we have $$u'_n-\alpha\ v'_n-\beta=(u_n-\alpha\ v_n-\beta)+(\alpha q_{n-1}-p_{n-1})$$
$$=\frac{(-1)^n(1-\Omega_n)}{\omega_n q_{n-1}+q_n}\,,$$
since $$\alpha q_{n-1}-p_{n-1}=q_{n-1}\left(\frac{\omega_n p_{n-1}+p_n}{\omega_n q_{n-1}+q_n}-\frac{p_{n-1}}{q_{n-1}}\right)=\frac{(-1)^n}{\omega_n q_{n-1}+q_n}\,.$$
Again, if we put $u''_{n+1}=p_n k_{n+1}+u_n$, $v''_{n+1}=q_n k_{n+1}+v_n$, we have
$$u''_n-\alpha\ v''_n-\beta=(u_{n-1}-\alpha\ v_{n-1}-\beta)+k_{n+1}(p_{n-1}-\alpha q_{n-1})$$
$$=\frac{(-1)^n\Omega_{n-1}+(-1)^{n-1}k_{n+1}}{\omega_{n-1}q_{n-2}+q_{n-1}}\,.$$

Consequently $|u_n-\alpha\ v_n-\beta|<1/q_n$, $|u'_n-\alpha\ v'_n-\beta|<1/q_n$; therefore if $1/q_m<\varepsilon$, then (u_n, v_n) (u'_n, v'_n) $(n\geqq m)$ are all the solutions of $|x-\alpha y-\beta|<\varepsilon$. Also for sufficiently large n, (u''_n, v''_n) is also a solution of $|x-\alpha y-\beta|<\varepsilon$, if $k_{n+1}<b_{n+1}$. The solutions (u''_n, v''_n) correspond to the intermediary convergents in the theory of continued fractions.