35. Operator-theoretical Treatment of Markoff's Process, II.

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§ 1. Let $P(x, E)$ denote the transition probability that the point x of the interval $\Omega=(0, 1)$ is transferred, by a simple Markoff's process, into the Borel set E of Ω after the elapse of a unit time. It is naturally assumed that $P(x, E)$ is completely additive for Borel sets E if x is fixed and that $P(x, E)$ is Borel measurable in x if E is fixed. $P(x, E)$ defines a linear operator P on the complex Banach space $(\mathfrak{M})$ in $(\mathfrak{M})$[1]:

$$P\cdot f=g\,, \quad g(E)=\int_{\Omega} P(x, E) f(dx)\,.$$

It is easy to see that the iterated operator P^n is defined by the kernel $P^{(n)}(x, E)=\int_{\Omega} P^{(n-1)}(x, dy)\, P(y, E) \left(P^{(1)}(x, E)=P(x, E)\right)$. In the preceding note,[2] it is proved that the following condition (D) implies the condition (K):

(D) $\begin{cases} \text{there exist an integer } s \text{ and positive constants } b, \eta\ (<1) \text{ such} \\ \text{that, if mes } (E)<\eta,\ P^{(s)}(x, E)<1-b \text{ uniformly in } x, E. \end{cases}$

(K) $\begin{cases} \text{there exist an integer } n \text{ and a completely continuous linear} \\ \text{operator } V \text{ such that } \| P^n-V\|_{\mathfrak{M}}<1. \end{cases}$

The condition (K) is more general than (D), since there exists $P(x, E)$ which satisfies (K) but not (D). In [I] it is proved that, if $P(x, E)$ satisfies (D), then

(B) $\begin{cases} \text{the proper values } \lambda \text{ with modulus 1 of } P \text{ are all roots of} \\ \text{unity.} \end{cases}$

Thus, combined with (K), we were able to give an operator-theoretical treatment of the Markoff's process $P(x, E)$ under the condition (D). (See [I].)

In the present note I intend to show that the condition (K) im-

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1) $(\mathfrak{M})$ is the linear space of all the totally additive set functions defined for all the Borel sets of Ω. For any $f\in(\mathfrak{M})$ we define its norm $\|f\|_{\mathfrak{M}}$ by the total variation of f on Ω.

2) K. Yosida: Operator-theoretical Treatment of the Markoff's Process, Proc. **14** (1938), 363. This note will be referred to as [I] below. It contains many misprints. On page 364, line 27 and line 28 $(\mathfrak{M})$ is to be read (M^*). On page 364, line 28 $h(dz)$ is to be read $h(z)$. On page 365, line 7 "$f_{i_k}(E)\cdot f_{j_k}(E)\equiv 0$ for $i\neq j$" is to be read "$f_{i_k}(E_{i_k})=1$ where $E_{i_k}\cdot E_{j_k}=$void for $i\neq j$." On page 367, line 4 and 5 "From... by (6)" is to be read "Evident from iii) and the equations $f_{(i+1)_a}(E)=\int_0^1 P(x, E) f_{i_a}(dx)$ below."

plies the property (B). Hence the results in [I] are, in essential, valid for the Markoff's process under the condition (K).

§ 2. Let $P(x, E)$ satisfy the condition (K). Then,[1] there exists completely continuous linear operators P_λ such that

$$(1)\qquad \begin{cases} \lim\limits_{n\to\infty} \left\| \frac{1}{n} \sum\limits_{m=1}^{n} \left(\frac{P}{\lambda}\right)^m - P_\lambda \right\|_{\mathfrak{M}} = 0\,, \\ P_\lambda^2 = P_\lambda\,, \quad PP_\lambda = P_\lambda P = \lambda P_\lambda \qquad (|\lambda|=1)\,. \end{cases}$$

P_λ is the projection operator which maps $(\mathfrak{M})$ on the proper space (Eigenraum) of P belonging to the proper value λ. Let P_λ be defined by the kernel $P_\lambda(x, E)$. We have, by $P^{(m)}(x, E) \geqq 0$ and $P^{(m)}(x, \Omega) \equiv 1$,

$$(2)\qquad P_1(x, E) \geqq 0\,, \quad P_1(x, \Omega) \equiv 1\,.$$

Thus $P_1 \neq 0$. Let $P \cdot f = f$, that is, $P_1 \cdot f = f$. Then, by (2), we obtain $P_1 \cdot \tilde{f} = \tilde{f}$, where $\tilde{f}(E)$ = the total variation of f on E. As P_1 is completely continuous, the number of the linearly independent solutions of $P_1 \cdot f = f$ is finite. Thus applying Kryloff-Bogoliouboff's arguments[2] we obtain the

Lemma. There exist $f_1, f_2, \dots, f_k \in (\mathfrak{M})$ with the properties:

$$P \cdot f_i = f_i\,, \quad f_i(E) \geqq 0\,, \quad f_i(E_i) = 1 \quad (E_i \cdot E_j = \text{void for } i \neq j)\,,$$

such that any f satisfying $P \cdot f = f$, $f(E) \geqq 0$, $f(\Omega) = 1$ is uniquely expressed as a linear combination $f(E) = \sum_{i=1}^{k} c_i f_i(E)$, $\sum_{i=1}^{k} c_i = 1$, $c_i \geqq 0$.

Hence, from $PP_1 = P_1$ and (2), we obtain

$$(3)\qquad \begin{cases} P_1(x, E) = \sum\limits_{i=1}^{k} c_i(x) f_i(E)\,, \\ c_i(x) \text{ measurable with } c_i(x) \geqq 0\,, \quad \sum\limits_{i=1}^{k} c_i(x) \equiv 1\,. \end{cases}$$

Let now $\lambda(|\lambda|=1)$ be a proper value of $P: P_\lambda \neq 0$. Let P_λ be defined by the kernel $P_\lambda(x, E)$. From $P_\lambda P^m = \lambda^m P_\lambda$ we see that the proper value equations

$$(4)\qquad \int_\Omega P^{(m)}(x, dy)\, g(y) = \lambda^m g(x) \qquad (m = 1, 2, \dots)$$

admit bounded measurable solution $g(x) \not\equiv 0$. We may assume that

$$(5)\qquad \| g \|_{M*} = \text{lowest upper bound}_{x \in \Omega} |g(x)| = 1\,.$$

Then we may prove that

$$(6)\qquad \text{there exists } x_0 \in \Omega \text{ such that } |g(x_0)| = 1\,.$$

1) K. Yosida: Abstract Integral Equations and the Homogeneous Stochastic Process, Proc. **14** (1938), 286. K. Yosida: Quasi-completely-continuous Linear Functional Operations, to appear soon in Jap. J. Math. Cf, also S. Kakutani: Iteration of Linear Operations in Complex Banach Spaces, Proc. **14** (1938), 292.

2) [I], Lemma 4.

Proof of (6). From (4) and $P^{(m)}(x, E) \geqq 0$, $P^{(m)}(x, \Omega) \equiv 1$, we obtain

$$|g(x)| \leqq \int_{|g(y)| \geq 1-\delta} P^{(m)}(x, dy) + (1-\delta) \int_{|g(y)|<1-\delta} P^{(m)}(x, dy) = 1 - \delta \int_{|g(y)|<1-\delta} P^{(m)}(x, dy),$$

$$(1 > \delta > 0).$$

Hence, by (1),

$$|g(x)| \leqq 1 - \delta \int_{|g(y)|<1-\delta} P_1(x, dy). \tag{7}$$

Let $x' \in \Omega$ be such that $|g(x')| \geqq 1 - \delta\varepsilon$ $(1 > \varepsilon > 0)$, then, by (7) and (3),

$$\begin{cases} \varepsilon \geqq P_1\big(x', E(\delta)\big) = \sum_{i=1}^{k} c_i(x') f_i\big(E_i - E_i \cdot E(\delta)\big), \\ E(\delta) = \underset{y}{E}\{|g(y)| \geqq 1 - \delta\}. \end{cases}$$

As $c_i(x) \geqq 0$, $\sum_{i=1}^{k} c_i(x) \equiv 1$ and ε, δ were arbitrary, we must have

$$f_i\big(E_i - E_i \cdot E(0)\big) = 0, \quad E(0) = \underset{y}{E}\{|g(y)| = 1\}$$

for a certain i (=1 or 2 or … or k). Thus, by $f_i(E_i) = 1$, $E(0)$ is not void. Q. E. D.

Next let $|g(x_0)| = 1$. Then, by (4) and $P^{(m)}(x_0, \Omega) = 1$, we have

$$\int_{\Omega} P^{(m)}(x_0, dy) \left\{1 - \frac{g(y)}{\lambda^m g(x_0)}\right\} = 0. \qquad (m = 1, 2, \ldots) \tag{8}$$

Put $g(y)/\lambda^m g(x_0) = h^{(m)}(y) = h_1^{(m)}(y) + \sqrt{-1}\, h_2^{(m)}(y)$, where $h_1^{(m)}(y)$ the real part of $h^{(m)}(y)$. From (5) we have $|h^{(m)}(y)| \leqq 1$. Thus $h_1^{(m)}(y) \leqq 1$, and if $h_1^{(m)}(y) = 1$ we must have $h^{(m)}(y) = 1$ viz. $g(y) = \lambda^m g(x_0)$.

From (8) we have

$$\int_{\Omega} P^{(m)}(x_0, dy) \big(1 - h_1^{(m)}(y)\big) = 0. \qquad (m = 1, 2, \ldots) \tag{9}$$

As $P^{(m)}(x_0, E) \geqq 0$, $P^{(m)}(x_0, \Omega) = 1$, $1 \geqq h_1^{(m)}(y)$ we must have

$$P^{(m)}\big(x_0, E(m)\big) = 1, \quad E(m) = \underset{y}{E}\{g(y) = \lambda^m g(x_0)\}.$$

Hence, if

(10) $E(i) \cdot E(j) \neq$ void for a certain couple of integers i, j with $i \neq j$,

then $\lambda^{i-j} = 1$, as was to be proved.

Now let (10) be not true. Then, by (1),

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} P^{(i)}\big(x_0, E(s)\big) = P_1\big(x_0, E(s)\big) = 0$$

for any s. Hence we would obtain $P_1\left(x_0, \sum_{s=1}^{\infty} E(s)\right) = \sum_{s=1}^{\infty} P_1\left(x_0, E(s)\right) = 0$. This is a contradiction, since, by (9) and (1),

$$P_1\left(x_0, \sum_{s=1}^{\infty} E(s)\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} P^{(i)}\left(x_0, \sum_{s=1}^{\infty} E(s)\right)$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} P^{(i)}\left(x_0, E(i)\right) = 1 .$$