PAPERS COMMUNICATED

30. *On the Compactness of a Class of Functions.*

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1. In this note we will consider the class of functions defined in the finite interval (a, b).

Let $\mathfrak{F}_c$ be a class of continuous functions defined in (a, b). If any sequence of functions in $\mathfrak{F}_c$ contains a uniformly convergent subsequence, then $\mathfrak{F}_c$ is called compact or compact in (C), where (C) denotes the class of all continuous functions. Arzèla's theorem concerning the compactness of $\mathfrak{F}_c$, is well known, which runs as follows:

Theorem A. *In order that the class $\mathfrak{F}_c$ be compact, it is necessary and sufficient that*

1°. $\mathfrak{F}_c$ is bounded, that is, there is a constant K such that $|f(x)| \leqq K$ for all f in $\mathfrak{F}_c$.

2°. $\mathfrak{F}_c$ is equally continuous, that is, for any positive number δ, there is an $\eta > 0$ such that the oscillation of functions in any interval with length less than η is less than δ.

Instead of (C) we take the class (L^p) $(p \geqq 1)$. Let $\mathfrak{F}_l$ be a class of functions in (L^p). If any sequence in $\mathfrak{F}_l$ contains a mean convergent subsequence with index p, then $\mathfrak{F}_l$ is called compact or compact in (L^p). Fréchet has proved the following theorem[1]:

Theorem B. *In order that the class $\mathfrak{F}_l$ be compact, it is necessary and sufficient that 1°. $\mathfrak{F}_l$ is almost equally continuous and 2°. $\mathfrak{F}_l$ is equally integrable.*

Finally let (S) be the class of all finite measurable functions. Let $\mathfrak{F}_s$ be a class of functions in (S). If any sequence in $\mathfrak{F}_s$ contains a subsequence convergent in measure, then $\mathfrak{F}_s$ is called compact or compact in (S). Fréchet has also proved that[2]

Theorem C. *In order that the class $\mathfrak{F}_s$ be compact, it is necessary and sufficient that 1°. $\mathfrak{F}_s$ is almost equally bounded and 2°. $\mathfrak{F}_s$ is almost equally continuous.*

On the other hand Kolmogoroff[3] has proved the following theorem:

Theorem D. *In order that $\mathfrak{F}_l$ in (L^p) be compact, it is necessary and sufficient that*

1°. $\mathfrak{F}_l$ is bounded, that is, there is a constant K such that

$$\int_a^b |f(x)|^p dx \leqq K$$

for all f in $\mathfrak{F}_l$.

1) M. Fréchet, Acta de Szeged, **8** (1937).

2) Fréchet, Fund. Math., **9** (1911).

3) A. Kolmogoroff, Göttinger Nachr., 1931. For the detailed literature, see T. Takahashi, Studia Math. **5** (1935).

2°. *For any* $\eta > 0$, *there is an* N_0 *such that*

$$\int_a^b |f(x) - f^N(x)|^p dx \leqq \eta$$

for all $N \geqq N_0$ *and for all* f *in* $\mathfrak{F}_l$, *where*

$$f^N(x) = f(x) \quad \text{if} \quad |f(x)| \leqq N; \quad f^N(x) = 0, \quad \text{otherwise.}$$

3°. *For any* $\eta > 0$, *there is a* δ_0 *such that*

$$\int_a^b |f(x) - f_\delta(x)|^p dx \leqq \eta$$

for all positive $\delta \leqq \delta_0$ *and for all* f *in* $\mathfrak{F}_l$, *where* $f_\delta(x) = \frac{1}{\delta}\int_0^\delta f(x+t)dt$.

The object of this paper is to prove two theorems concerning the compactness of $\mathfrak{F}_c$ and $\mathfrak{F}_s$, the condition being of the type of Theorem D.

2. Theorem 1. *In order that* $\mathfrak{F}_c$ *in* (C) *be compact, it is necessary and sufficient that 1°.* $\mathfrak{F}_c$ *is bounded and 2°.*

$$\lim_{\delta \to 0} \frac{1}{\delta}\int_0^\delta f(x+t)\, dt = f(x) \tag{1}$$

uniformly for all x *and all* f *in* $\mathfrak{F}_c$.

Let us suppose that $\mathfrak{F}_c$ is compact. The condition 1° is evidently necessary. If $\mathfrak{F}_c$ is compact, then $\mathfrak{F}_c$ is equally continuous and then the condition 2° follows.

Conversely, suppose that the conditions in the theorem be satisfied. Let $\{f_n(x)\}$ be any sequence in $\mathfrak{F}_c$. By 1°

$$\int_a^b |f_n(x)|^p dx \leqq K.$$

By 2°, for any $\eta > 0$, there is a δ, such that

$$\int_a^b \left|f_n(x) - \big(f_n(x)\big)\delta\right| dx < \eta \qquad (n = 1, 2, \ldots)$$

for all $\delta < \delta_0$. By Theorem D, $\{f_n(x)\}$ contains a subsequence $\{f_{n_K}(x)\}$ which converges in mean, that is, there is a function $f(x)$ such that

$$\lim_{K \to \infty} \int_a^b |f_{n_K}(x) - f(x)|\, dx = 0. \tag{2}$$

On the other hand

$$\left|f_{n_K}(x) - f(x)\right| \leqq \left|f_{n_K}(x) - \big(f_{n_K}(x)\big)\delta\right| + \left|\big(f_{n_K}(x)\big)\delta - \big(f(x)\big)\delta\right| + \left|\big(f(x)\big)\delta - f(x)\right|.$$

For any $\eta > 0$, we can take δ_0 such that

$$\left|f(x) - \big(f(x)\big)\delta_0\right| < \eta, \quad \left|f_{n_K}(x) - \big(f_{n_K}(x)\big)\delta_0\right| < \eta \qquad (K = 1, 2, \ldots).$$

$$\left|\left(f_{n_K}(x)\right)\delta_0-\left(f(x)\right)\delta\right|=\frac{1}{\delta}\left|\int_0^\delta\{f_{n_K}(x+t)-f(x+t)\}\,dt\right|$$
$$\leqq\frac{1}{\delta}\int_0^\delta|f_{n_K}(x+t)-f(x+t)|\,dt\leqq\frac{1}{\delta}\int_a^b|f_{n_K}(t)-f(t)|\,dt,$$

which tends to zero as $K\to\infty$ by (2). Therefore $\{f_{n_K}(x)\}$ is uniformly convergent. Thus the conditions 1° and 2° are sufficient.

3. Theorem A can be deduced from Theorem 1. The direct proof of Theorem 1 can be done by the method used in proving Theorem A. Theorem D is proved by the use of Theorem A. Above consideration shows that Theorem A is proved by the use of Theorem D.

Let (M) be the class of all bounded measurable functions. When the class $\mathfrak{F}_m$ in (M) is considered instead of $\mathfrak{F}_c$, we get the following theorem:

Theorem 2. *In order that any sequence in the bounded class $\mathfrak{F}_m$ contains a uniformly convergent subsequence, it is necessary and sufficient that (1) holds uniformly for all x and for all f in $\mathfrak{F}_m$.*

This is the analogy of the Veress's theorem.

In this theorem, if we replace uniform convergence by the almost everywhere uniform convergence, then the necessary and sufficient condition becomes that (1) holds almost everywhere uniformly.

4. Let us consider the class (S). In (S) we introduce the metric due to Fréchet such that

$$|f|=\int_a^b\frac{|f(t)|}{1+|f(t)|}\,dt.$$

Theorem 3. *In order that the class $\mathfrak{F}_s$ is compact, it is necessary and sufficient that*

1°. For any $\delta>0$ and $N>0$, there exists an M such that

$$\frac{1}{\delta}\left|\int_0^\delta\{f(x+t)\}^N\,dt\right|\leqq M$$

for all x and for all f in $\mathfrak{F}_m$.

2°. For any $\eta>0$, there exists N_0 such that $|f-f_\delta^N|\leqq\eta$ for all $N\geqq N_1$.

3°. For any $\eta>0$, there exist N_1 and δ_1 such that $|f-f_\delta^N|\leqq\eta$ for all $N\geqq N_1$ and for all $\delta\leqq\delta_1$.

Proof is done similarly as that of Theorem D.