PAPERS COMMUNICATED

43. *Birkhoff's Ergodic Theorem and the Maximal Ergodic Theorem.*

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1. *Statement of the theorem.* Let S be a space in which a measure of Lebesgue type is defined, and let T be a one-to-one measure-preserving transformation of S into itself. We do not assume that the total measure mes (S) is finite. For any real valued function $f(x)$ defined on S, we define the functions $\bar{f}(x), \underline{f}(x), f^*(x)$ and $f_*(x)$ as follows:

$$\begin{cases} \bar{f}(x) = \overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x), & \underline{f}(x) = \underline{\lim}_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x), \\ f^*(x) = \underset{0\leq n<\infty}{\text{l. u. b.}} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x), & f_*(x) = \underset{0\leq n<\infty}{\text{g. l. b.}} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x). \end{cases}$$

If $f(x)$ is measurable and absolutely integrable on S, then we can prove the following two theorems:

Theorem 1. For any pair of real numbers α and β, we have

$$(1) \quad \begin{cases} \alpha \text{ mes } \big(E(\alpha, \beta)\big) \leqq \int_{E(\alpha,\beta)} f(x)\, dx \leqq \beta \text{ mes } \big(E(\alpha, \beta)\big), \\ \text{where } \quad E(\alpha, \beta) = \underset{x}{E} [\bar{f}(x) > \alpha, \underline{f}(x) < \beta]. \end{cases}$$

Consequently, $\alpha > \beta$ implies mes $\big(E(\alpha, \beta)\big) = 0$, and since this is true for any pair of real numbers α and β with $\alpha > \beta$, we have $\bar{f}(x) = \underline{f}(x)$ almost everywhere; that is,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) = f_1(x)$$

exists almost everywhere.

Theorem 2. For any real number α we have

$$(2) \quad \begin{cases} \alpha \text{ mes } \big(E^*(\alpha)\big) \leqq \int_{E^*(\alpha)} f(x)\, dx, \quad \alpha \text{ mes } \big(E_*(\alpha)\big) \geqq \int_{E_*(\alpha)} f(x)\, dx, \\ \text{where } \quad E^*(\alpha) = \underset{x}{E} [f^*(x) > \alpha] \text{ and } E_*(\alpha) = \underset{x}{E} [f_*(x) < \alpha]. \end{cases}$$

Theorem 1 is the *Ergodic Theorem of Birkhoff* in its form given by A. Kolmogoroff.[1] Theorem 2 is new. We shall call Theorem 2

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1) A. Kolmogoroff: Ein vereinfachter Beweis des Birkhoff-Khintchinschen Ergodensatzes, Recueil Math., **44** (1937), 366-368. See also E. Hopf: Ergodentheorie, Ergebnisse der Math., Heft **5** (1937).

the *Maximal Ergodic Theorem.* Recently N. Wiener[2] obtained the analogous result:

$$(2') \quad \begin{cases} \text{if } \operatorname{mes}(S)=\text{finite, and if } f(x)\geqq 0 \text{ throughout on } S, \\ \text{then } \alpha \operatorname{mes}\big(E^*(\alpha)\big) \leqq \int_S f(x)\,dx. \end{cases}$$

This result is clearly weaker than (2). Wiener's proof of (2′) is based on the so-called Maximal Theorem of Hardy and Littlewood[3]; and using (2′) he deduced from the Mean Ergodic Theorem of v. Neumann a new proof of the Ergodic Theorem of Birkhoff. Wiener has also obtained from (2′) the so-called *Dominated Ergodic Theorem.*[4] It is to be noted that the latter is also possible even if we have no assumption that mes (S)=finite, while the former is not always possible without this assumption.

In the present note, we shall give a direct proof of Theorem 2. Our method of proof is a modification of that of Khintchine-Kolmogoroff,[1] which was used to prove Theorem 1; and it is to be noted that we can prove Theorem 1 (Birkhoff's Ergodic Theorem) and Theorem 2 (Maximal Ergodic Theorem) simultaneously by the same principle without appealing to Maximal Theorem nor to the Mean Ergodic Theorem.

2. *Proof of Theorem 2.* We define

$$(3) \qquad f_{ab}(x)=\frac{1}{b-a}\sum_{i=a}^{b-1}f(T^i x), \qquad a<b.$$

For any fixed $x\in S$, consider the pair of integers a and b such that $f_{ab}(x)>\alpha$ while $f_{ab'}(x)\leqq \alpha$ for any b' with $a<b'<b$. Such an interval (a, b) is called a *maximal interval* (corresponding to α and x), and $b-a$ is called the *length* of this maximal interval. Of two maximal intervals (a, b) and (a', b') (both corresponding to α and x), the one may contain the other; but these cannot overlap each other. For, if $a<a'<b<b'$, we have

$$f_{ab}(x)=\frac{(a'-a)\cdot f_{aa'}(x)+(b-a')\cdot f_{a'b}(x)}{b-a}$$

and, since $f_{aa'}(x)\leqq \alpha$ and $f_{a'b}(x)\leqq \alpha$ by assumption, we have $f_{ab}(x)\leqq \alpha$, contrary to the assumption that (a, b) is maximal. A maximal interval (a, b) (corresponding to α and x) of length $b-a\leqq s$ is called *s-maximal* if it is contained in no other maximal interval (corresponding to α and to x) of length $\leqq s$. Thus all s-maxinal intervals (corresponding to α and to x) lie outside each other.

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2) N. Wiener: The Ergodic Theorem, Duke Math., Journ., **5** (1939), 1-18.

3) G. H. Hardy, J. E. Littlewood and G. Pólya: Inequalities. Cambridge (1935).

4) N. Wiener: The Homogeneous Chaos, Amer. Journ. of Math., **60** (1938), 897-936. In this paper Zygmund's class only was considered. The general case $L^p(p>1)$ was obtained by N. Wiener and M. Fukamiya independently. N. Wiener: the paper cited in the footnote (2). M. Fukamiya: On Dominated Ergodic Theorem in $L^p(p\geqq 1)$, to be published in Tôhoku Math. Journ. Fukamiya's proof also appeals to the Maximal Theorem of Hardy and Littlewood.

Now let $E_s^*(\alpha)$ be the set of all the points $x \in S$ such that there exists an s-maximal interval (a, b) (corresponding to α and to x) with $a \leqq 0 < b$. It is clear, by the argument above, that to any point $x \in E_s^*(\alpha)$ there corresponds one and only one s-maximal interval of this sort. Since $f_{ab}(x) > \alpha$ and since $f_{ab'}(x) \leqq \alpha$ for any b' with $a < b' < b$, we must have $f_{ob}(x) > \alpha$ and consequently $E_s^*(\alpha) \subset E^*(\alpha)$ for any s. Moreover, by the definition of $E^*(\alpha)$, we have

$$\lim_{s \to \infty} E_s^*(\alpha) = E^*(\alpha). \tag{4}$$

On the other hand, $E_s^*(\alpha)$ may be divided into disjoint subsets $E_{pq}^*(\alpha)$:

$$E_s^*(\alpha) = \sum_{q=1}^{s} \sum_{p=0}^{q-1} E_{pq}^*(\alpha), \tag{5}$$

where $E_{pq}^*(\alpha)$, $0 \leqq p < q \leqq s$, is the set of all the points $x \in E_s^*(\alpha)$ whose corresponding s-maximal interval is $(-p, -p+q)$. From the identity:

$$\frac{1}{q} \sum_{i=-p}^{-p+q-1} f(T^i x) = \frac{1}{q} \sum_{i=0}^{q-1} f(T^i T^{-p} x)$$

we see that

$$T^{-p} E_{pq}^*(\alpha) = E_{oq}^*(\alpha),$$

and, since T is measure-preserving, we have

$$\begin{cases} \text{mes}\left(E_{pq}^*(\alpha)\right) = \text{mes}\left(E_{oq}^*(\alpha)\right), \\ \displaystyle\int_{E_{pq}^*(\alpha)} f(x)\,dx = \int_{E_{oq}^*(\alpha)} f(T^p x)\,dx. \end{cases} \tag{6}$$

Consequently, we have by (5) and (6)

$$\begin{cases} \displaystyle\int_{E_s^*(\alpha)} f(x)dx = \sum_{q=1}^{s}\sum_{p=0}^{q-1} \int_{E_{pq}^*(\alpha)} f(x)dx = \sum_{q=1}^{s}\sum_{p=0}^{q-1} \int_{E_{oq}^*(\alpha)} f(T^p x)dx = \sum_{q=1}^{s} \int_{E_{oq}^*(\alpha)} q \cdot f_{oq}(x)dx \\ \displaystyle\geqq \sum_{q=1}^{s} \int_{E_{oq}^*(\alpha)} q \cdot \alpha\, dx = \sum_{q=1}^{s} q \cdot \alpha \text{ mes}\left(E_{oq}^*(\alpha)\right) = \alpha \sum_{q=1}^{s}\sum_{p=0}^{q-1} \text{mes}\left(E_{pq}^*(\alpha)\right) \\ \displaystyle\qquad = \alpha \text{ mes}\left(E_s^*(\alpha)\right). \end{cases} \tag{7}$$

Hence, by (4), we obtain

$$\int_{E^*(\alpha)} f(x)dx \geqq \alpha \text{ mes}\left(E^*(\alpha)\right).$$

Thus the first part of Theorem 2 is proved, and the second part may be proved analogously. We may also obtain the proof of Theorem 1, if we start from $E(\alpha, \beta)$ instead of from $E^*(\alpha)$, and if we consider $E_s(\alpha, \beta) = E_s^*(\alpha) \cdot E(\alpha, \beta)$ and $E_{pq}(\alpha, \beta) = E_{pq}^*(\alpha) \cdot E(\alpha, \beta)$ instead of $E_s^*(\alpha)$ and $E_{pq}^*(\alpha)$ respectively, remembering the invariance of $E(\alpha, \beta)$: $E(\alpha, \beta) = TE(\alpha, \beta)$. This is indeed the proof of Theorem 1 due to A. Kolmogoroff.

3. *Integrability of the functions $f_1(x)$, $f^*(x)$ and $f_*(x)$.* We can see easily from Theorem 1 that, if $f(x)$ is absolutely integrable, the limit function $f_1(x)$ $(=\bar{f}(x)=\underline{f}(x)$ almost everywhere) is also absolutely integrable. In order to show this, it is sufficient to consider the case that $f(x) \geqq 0$ throughout on S. Denoting again by $E(\alpha, \beta)$ the set of all the points $x \in S$ such that $\alpha < f_1(x) < \beta$, we have for any pair of real numbers α and β with $0 < \alpha < \beta$

$$\alpha \text{ mes } \big(E(\alpha, \beta)\big) \leqq \int_{E(\alpha, \beta)} f(x)\,dx \leqq \beta \text{ mes } \big(E(\alpha, \beta)\big)\,,$$

and, since mes $\big(E(\alpha, \beta)\big) < \infty$, we have

$$\int_{E(\alpha, \beta)} f_1(x)\,dx = \int_{E(\alpha, \beta)} f(x)\,dx\,.$$

Since α and β $(0 < \alpha < \beta)$ are arbitrary, we have

$$(8) \qquad \int_S f_1(x)\,dx = \int_{E(0, \infty)} f_1(x)\,dx = \int_{E(0, \infty)} f(x)\,dx \leqq \int_S f(x)\,dx\,.$$

Thus we have proved that $f_1(x)$ is absolutely integrable with the additional inequality (8).

If, moreover, $f(x)$ belongs to the Lebesgue's class L^p $(p > 1)$, then $f^*(x)$ and $f_*(x)$ belong also to the same class L^p; and if $f(x)$ belongs to the Zygmund's class:

$$\int_S |f(x)| \overset{+}{\log} |f(x)|\,dx = \text{finite},$$

then $f^*(x)$ and $f_*(x)$ both belong to the class L^1. These results (Dominated Ergodic Theorem) were obtained from (2') by N. Wiener, and directly from the Maximal Theorem of Hardy and Littlewood by M. Fukamiya, in case mes(S) = finite. The same argument as that used by Wiener will lead us to the same conclusion for the class L^p $(p > 1)$ even in the general case mes $(S) = \infty$ from our (2); for, the assumption that mes(S) = finite is not needed in this part of the proof of Wiener's. We therefore omit the proof.