20. Conformally Separable Quadratic Differential Forms.

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1. Let us consider an n-dimensional Riemannian space whose first fundamental form is

$$ds^2 = g_{\mu\nu} du^\mu du^\nu, \qquad (\lambda, \mu, \nu, \dots = 1, 2, 3, \dots, n). \tag{1.1}$$

In this space, the equations $u^a = \text{const.}$ $(a, b, c, \dots = 1, 2, \dots, m;\ m < n)$ define a family of $(n-m)$-dimensional subspaces V_{n-m}, and the equations $u^i = \text{const.}$ $(h, i, j, \dots = m+1, m+2, \dots, n)$ a family of m-dimensional subspaces V_m in V_n.

It has been shown by E. Bompiani[1] that the necessary and sufficient condition that the subspaces V_{n-m} and the subspaces V_m orthogonal to V_{n-m} be totally geodesic in V_n is that

$$g_{ab} = f_{ab}(u^c), \qquad g_{jk} = f_{jk}(u^i), \qquad g_{ai} = 0, \tag{1.2}$$

and consequently the first fundamental form (1.1) may be written in the form

$$ds^2 = f_{ab}(u^c) du^a du^b + f_{jk}(u^i) du^j du^k. \tag{1.3}$$

In this case, the quadratic differential form (1.1) is said to be separable, and $f_{ab}(u^c)du^a du^b$ and $f_{jk}(u^i)du^j du^k$ are called the components of the separable quadratic differential form (1.1).

Recently, A. Fialkow[2] has proved that if the first fundamental form of an Einstein space of dimensionality $n > 3$ of mean curvature a is separable into two components whose dimensions exceed 1, then each component is also the first fundamental form of an Einstein space of mean curvature a.

In the present Note, we try to find the necessary and sufficient condition that the subspaces V_{n-m} and the subspaces V_m orthogonal to V_{n-m} be both totally umbilical in V_n, and to obtain a theorem corresponding to the theorem of A. Fialkow quoted above.

2. We assume that the subspaces V_{n-m} and the subspaces V_m orthogonal to V_{n-m} be both totally umbilical $(n-m,\ m \geqq 2)$ in V_n. The orthogonality between V_{n-m} and V_m gives us immediately

$$g_{ai} = 0, \qquad g^{ai} = 0. \tag{2.1}$$

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1) E. Bompiani, Spazi Riemanniani luoghi di varieta totalmente geodetiche, Rendiconti del Circolo Matematico di Palermo, **48** (1924) p. 124.

2) A. Fialkow, Totally geodesic Einstein spaces, Bulletin of the American Mathematical Society, **45** (1939) p. 423.

The equations defining V_{n-m} being $u^a=\text{const.}$, we have for V_{n-m}

$$ds^2=\bar{g}_{jk}du^j du^k, \qquad (\bar{g}_{jk}=g_{jk}) \tag{2.2}$$

$$B_j^{\cdot\lambda}=\frac{\partial u^\lambda}{\partial u^j}=\delta_j^\lambda, \tag{2.3}$$

so that the fundamental tensors of V_{n-m} are given by the equations

$$\bar{g}_{jk}=g_{jk}, \qquad \bar{g}^{jk}=g^{jk},$$

and the Christoffel symbols by

$$\overline{\left\{{}^{\,i}_{jk}\right\}}=\frac{1}{2}\bar{g}^{ih}\left(\frac{\partial \bar{g}_{hj}}{\partial u^k}+\frac{\partial \bar{g}_{hk}}{\partial u^j}-\frac{\partial \bar{g}_{jk}}{\partial u^h}\right) \tag{2.4}$$
$$=\frac{1}{2}g^{i\lambda}\left(\frac{\partial g_{\lambda j}}{\partial u^k}+\frac{\partial g_{\lambda k}}{\partial u^j}-\frac{\partial g_{jk}}{\partial u^\lambda}\right)$$
$$=\left\{{}^{\,i}_{jk}\right\}$$

where we have denoted by $\left\{{}^{\,\lambda}_{\mu\nu}\right\}$ the Christoffel symbols formed with $g_{\mu\nu}$.

We shall now calculate the conformal tensor $M_{jk}^{\cdot\cdot\lambda}$ [1] for V_{n-m}. The tensor $H_{jk}^{\cdot\cdot\lambda}$ being defined by

$$H_{jk}^{\cdot\cdot\lambda}=B_{j;k}^{\cdot\lambda}=\frac{\partial B_j^{\cdot\lambda}}{\partial u^k}+B_j^{\cdot\mu}\left\{{}^{\,\lambda}_{\mu\nu}\right\}B_k^{\cdot\nu}-B_i^{\cdot\lambda}\overline{\left\{{}^{\,i}_{jk}\right\}}, \tag{2.5}$$

the equations (2.3), (2.4) and (2.5) give us

$$H_{jk}^{\cdot\cdot\lambda}=\left\{{}^{\,\lambda}_{jk}\right\}-\delta_i^\lambda\left\{{}^{\,i}_{jk}\right\}. \tag{2.6}$$

Substituting (2.6) in the equation

$$M_{jk}^{\cdot\cdot\lambda}=H_{jk}^{\cdot\cdot\lambda}-\frac{1}{n-m}\bar{g}^{ih}H_{ih}^{\cdot\cdot\lambda}\bar{g}_{jk},$$

we have

$$M_{jk}^{\cdot\cdot\lambda}=\left\{{}^{\,\lambda}_{jk}\right\}-\delta_i^\lambda\left\{{}^{\,i}_{jk}\right\}-\frac{1}{n-m}\bar{g}^{ih}\left[\left\{{}^{\,\lambda}_{ih}\right\}-\delta_l^\lambda\left\{{}^{\,l}_{ih}\right\}\right]\bar{g}_{jk}. \tag{2.7}$$

If the subspaces V_{n-m} are totally umbilical, the tensor $M_{jk}^{\cdot\cdot\lambda}$ must be identically zero, but for $\lambda=m$ the equation (2.7) gives us $M_{jk}^{\cdot\cdot m}=0$.

Putting $\lambda=a$, we have from (2.7)

$$\left\{{}^{\,a}_{jk}\right\}-\frac{1}{n-m}\bar{g}^{ih}\left\{{}^{\,a}_{ih}\right\}\bar{g}_{jk}=0. \tag{2.8}$$

On the other hand, we have

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1) For the notation adopted here, see K. Yano, Sur les équations de Gauss dans la géométrie conforme des espaces de Riemann, Proc. **15** (1939), 249, and Sur les équations de Codazzi dans la géométrie conforme des espaces de Riemann, Proc. **15** (1939), 340.

$$\left\{ {a \atop jk} \right\} = \frac{1}{2} g^{a\lambda} \left(\frac{\partial g_{\lambda j}}{\partial u^k} + \frac{\partial g_{\lambda k}}{\partial u^j} - \frac{\partial g_{jk}}{\partial u^\lambda} \right)$$

$$= -\frac{1}{2} g^{ab} \frac{\partial g_{jk}}{\partial u^b},$$

consequently (2.8) becomes

$$-\frac{1}{2} g^{ab} \frac{\partial g_{jk}}{\partial u^b} + \frac{1}{2(n-m)} g^{ih} g^{ab} \frac{\partial g_{ih}}{\partial u^b} g_{jk} = 0,$$

that is to say

$$\frac{\partial g_{jk}}{\partial u^a} = \frac{1}{n-m} g^{ih} \frac{\partial g_{ih}}{\partial u^a} g_{jk},$$

$$\frac{\partial g_{jk}}{\partial u^a} = \frac{\partial \log |g_{ih}|^{\frac{1}{n-m}}}{\partial u^a} g_{jk},$$

$|g_{ih}|$ being the determinant formed with g_{ih}.

These equations show that the functions g_{jk} must have the form

$$g_{jk} = \sigma(u^\lambda) f_{jk}(u^i), \tag{2.9}$$

and if the condition (2.9) is satisfied, it is easy to verify that the subspaces V_{n-m} are totally umbilical in V_n. Similarly, if the subspaces V_m are also totally umbilical in V_n, we find that the functions g_{ab} must have the form

$$g_{ab} = \rho(u^\lambda) f_{ab}(u^c). \tag{2.10}$$

Then we have the theorem:

The necessary and sufficient condition that the subspaces V_{n-m} and V_m orthogonal to V_{n-m} be totally umbilical in V_n ($n-m \geqq 2$, $m \geqq 2$) is that the first fundamental form of V_n has the following form:

$$ds^2 = \rho(u^\lambda) f_{ab}(u^c) du^a du^b + \sigma(u^\lambda) f_{ij}(u^k) du^i du^j. \tag{2.11}$$

In this case, we shall call the quadratic form (1.1) conformally separable form and $\rho(u^\lambda) f_{ab}(u^c) du^a du^b$ and $\sigma(u^\lambda) f_{ij}(u^k) du^i du^j$ its components.

3. A space of constant curvature is defined as a Riemann space V_n whose curvature tensor $R^{\lambda}_{\cdot\mu\nu\omega}$ has the form

$$R^{\lambda}_{\cdot\mu\nu\omega} = \frac{R}{n(n-1)} (g_{\mu\nu} \delta^\lambda_\omega - g_{\mu\omega} \delta^\lambda_\nu), \tag{3.1}$$

it is well known that the scalar R is constant throughout the space.

We shall consider, in this Paragraph, two orthogonal families of totally umbilical subspaces V_{n-m} and V_m in such a space of constant curvature.

The first fundamental quadratic differential form being

$$ds^2 = \rho(u^\lambda) f_{ab}(u^c) du^a du^b + \sigma(u^\lambda) f_{ij}(u^k) du^i du^j, \tag{3.2}$$

we have the following expression for the Christoffel symbols $\{^{\lambda}_{\mu\nu}\}$ of V_n:

$$
(3.3)\quad \begin{cases}
\{^a_{bc}\} = \frac{1}{2} g^{a\lambda}\left(\frac{\partial g_{\lambda b}}{\partial u^c} + \frac{\partial g_{\lambda c}}{\partial u^b} - \frac{\partial g_{bc}}{\partial u^\lambda}\right) = \overline{\{^a_{bc}\}}, \\
\{^i_{bc}\} = \frac{1}{2} g^{i\lambda}\left(\frac{\partial g_{\lambda b}}{\partial u^c} + \frac{\partial g_{\lambda c}}{\partial u^b} - \frac{\partial g_{bc}}{\partial u^\lambda}\right) = -\bar{g}^{ij}\rho_j \bar{g}_{bc}, \\
\{^a_{ic}\} = \frac{1}{2} g^{a\lambda}\left(\frac{\partial g_{\lambda i}}{\partial u^c} + \frac{\partial g_{\lambda c}}{\partial u^i} - \frac{\partial g_{ic}}{\partial u^\lambda}\right) = \rho_i \delta^a_c, \\
\{^a_{ij}\} = \frac{1}{2} g^{a\lambda}\left(\frac{\partial g_{\lambda i}}{\partial u^j} + \frac{\partial g_{\lambda j}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^\lambda}\right) = -\bar{g}^{ab}\sigma_b \bar{g}_{ij}, \\
\{^i_{jc}\} = \frac{1}{2} g^{i\lambda}\left(\frac{\partial g_{\lambda j}}{\partial u^c} + \frac{\partial g_{\lambda c}}{\partial u^j} - \frac{\partial g_{jc}}{\partial u^\lambda}\right) = \sigma_c \delta^i_j, \\
\{^i_{jk}\} = \frac{1}{2} g^{i\lambda}\left(\frac{\partial g_{\lambda j}}{\partial u^k} + \frac{\partial g_{\lambda k}}{\partial u^j} - \frac{\partial g_{jk}}{\partial u^\lambda}\right) = \overline{\{^i_{jk}\}},
\end{cases}
$$

where

$$
(3.4)\quad \rho_j = \frac{1}{2}\frac{\partial \log \rho}{\partial u^j}, \qquad \sigma_b = \frac{1}{2}\frac{\partial \log \sigma}{\partial u^b},
$$

and $\overline{\{^a_{bc}\}}$ and $\overline{\{^i_{jk}\}}$ represent the Christoffel symbols for V_{n-m} and V_m respectively.

The curvature tensor $R^{\lambda}_{\cdot\mu\nu\omega}$ being defined by

$$
(3.5)\quad R^{\lambda}_{\cdot\mu\nu\omega} = \frac{\partial \{^{\lambda}_{\mu\nu}\}}{\partial u^\omega} - \frac{\partial \{^{\lambda}_{\mu\omega}\}}{\partial u^\nu} + \{^{\alpha}_{\mu\nu}\}\{^{\lambda}_{\alpha\omega}\} - \{^{\alpha}_{\mu\omega}\}\{^{\lambda}_{\alpha\nu}\},
$$

we have

$$
R^a_{\cdot bcd} = \frac{\partial \{^a_{bc}\}}{\partial u^d} - \frac{\partial \{^a_{bd}\}}{\partial u^c} + \{^e_{bc}\}\{^a_{ed}\} - \{^e_{bd}\}\{^a_{ec}\} + \{^i_{bc}\}\{^a_{id}\} - \{^i_{bd}\}\{^a_{ic}\}.
$$

Denoting by $\bar{R}^a_{\cdot bcd}$ the curvature tensor of V_{n-m}, we obtain

$$
R^a_{\cdot bcd} = \bar{R}^a_{\cdot bcd} - \bar{g}^{ij}\rho_j \bar{g}_{bc}\rho_i \delta^a_d + \bar{g}^{ij}\rho_j \bar{g}_{bd}\rho_i \delta^a_c,
$$

hence

$$
(3.6)\quad R^a_{\cdot bcd} = \bar{R}^a_{\cdot bcd} - \bar{g}^{ij}\rho_i\rho_j(\bar{g}_{bc}\delta^a_d - \bar{g}_{bd}\delta^a_c).
$$

Substituting the equation (3.6) into (3.1), we have

$$
(3.7)\quad \bar{R}^a_{\cdot bcd} = \left[\frac{R}{n(n-1)} + \bar{g}^{ij}\rho_i\rho_j\right](\bar{g}_{bc}\delta^a_d - \bar{g}_{bd}\delta^a_c).
$$

We can prove also for V_m the following equation

$$
(3.8)\quad \bar{R}^i_{\cdot jkh} = \left[\frac{R}{n(n-1)} + \bar{g}^{ab}\sigma_a\sigma_b\right](\bar{g}_{jk}\delta^i_h - \bar{g}_{jh}\delta^i_k).
$$

(3.7) and (3.8) prove the theorem:

If the first fundamental quadratic differential form of a Riemann space of constant curvature is conformally separable into two components, then each component is also the first fundamental form of a Riemann space of constant curvature.