17. *Some Theorems on Abstractly-valued Functions in an Abstract Space.*

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1. *Introduction and Theorems.* Let $f(t)$ be an abstractly-valued function defined on $[0,1]$ whose range lies in a Banach space $\mathfrak{X}$. Under $L^p(\mathfrak{X})$ $(p \geqq 1)$ $\big(L^1(\mathfrak{X}) = L(\mathfrak{X})\big)$ we understand the class of all functions $f(t)$ measurable in the sense of S. Bochner such that $\int_0^1 \|f(t)\|^p dt < \infty$. $L^p(\mathfrak{X})$ $(p \geqq 1)$ is a Banach space with $\|f\| = \Big(\int_0^1 \|f(t)\|^p dt\Big)^{\frac{1}{p}}$ as its norm.

The purpose of the present note is to prove the following theorems:

Theorem 1. In an arbitrary space T let $\mathfrak{E}$ be a Borel family of subsets that includes T, and $\alpha(E)$ be a non-negative set function which is completely additive over $\mathfrak{E}$. If an abstractly-valued function $X(E)$, defined from $\mathfrak{E}$ to a Banach space $\mathfrak{X}$, is weakly absolutely continuous (i.e., for each φ in $\bar{\mathfrak{X}}$, the numerical function $\varphi X(E)$ is completely additive and absolutely continuous), then $X(E)$ is even strongly absolutely continuous (i.e., $X(E)$ is strongly completely additive, and for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|X(E)\| < \varepsilon$ whenever $\alpha(E) < \delta$).

Theorem 2. If $\mathfrak{X}$ is locally weakly compact, and if a sequence $\{f_n(t)\}$ $(n=1,2,\dots)$ of elements of $L(\mathfrak{X})$ is equi-integrable, then $\{f_n(t)\}$ $(n=1,2,\dots)$ contains a subsequence which converges weakly (as a sequence in $L(\mathfrak{X})$) to an element $f(t) \in L(\mathfrak{X})$.

Theorem 3. If $\mathfrak{X}$ is locally weakly compact, then $L^p(\mathfrak{X})$ $(p > 1)$ is also locally weakly compact.

Theorem 4. If $\mathfrak{X}$ is locally weakly compact, then $L(\mathfrak{X})$ is weakly complete.

Theorem 4 is a generalization of a result of S. Bochner-A. E. Taylor,[1] who assumed that $\mathfrak{X}$ is reflexive and that $\mathfrak{X}$ and $\bar{\mathfrak{X}}$ both satisfy the condition (D). Theorem 2[2] is an analogue of H. Lebesgue's theorem,[3] which is concerned with numerical-valued functions. These two theorems will be proved by using Theorem 1, and this theorem was announced without proof by B. J. Pettis[4] under the additional assumption[5] that T is expressible in the form: $T = \sum_{i=1}^{\infty} T_i$ with $\alpha(T_i) < \infty$, $i = 1, 2, \dots$.

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1) S. Bochner-A. E. Taylor: Linear functionals on certain spaces of abstractly-valued functions, Annals of Math., **39** (1938), 913–944. Theorem 5.2.

2) Theorem 2 may be considered as a precision to Theorem 4.2. (p. 923) in the paper of S. Bochner-A. E. Taylor cited in (1).

3) H. Lebesgue: Sur les intégrales singulières, Ann. de la Fac. des Sci. de Toulose, **1** (1909), especially p. 52.

4) B. J. Pettis: Bull. Amer. Math. Soc., (Abstracts), **44**-**2** (1939), 677.

5) This fact was suggested to me by K. Yosida.

The proofs of these theorems are much simplified by using the Vitali-Hahn-Saks' theorem.[6]

Lastly, Theorem 3, which extends the well-known theorem of F. Riesz concerning the Banach space (L^p) $(p>1)$ of numerical functions, seems to be new, although in case $\mathfrak{X}$ is separable this theorem follows directly from a result of S. Bochner-A. E. Taylor.[7]

2. *Proof of Theorem 1.*[8] Since $\varphi X(E)$ is completely additive for each φ in $\bar{\mathfrak{X}}$, for any sequence $\{E_n\}$ $(n=1, 2, \dots)$ of disjoint elements of $\mathfrak{E}$, $\sum_{n=1}^{\infty} X(E_n)$ is unconditionally convergent and $\sum_{n=1}^{\infty} X(E_n)=X(\sum_{n=1}^{\infty} E_n)$,[9] i. e., $X(E)$ is strongly completely additive over $\mathfrak{E}$.

Suppose now that $X(E)$ is not strongly absolutely continuous. Then for some $\varepsilon_0>0$ there exists a sequence $\{E_n\}$ $(n=1, 2, \dots)$ of elements of $\mathfrak{E}$ with $\lim_{n\to\infty} \alpha(E_n)=0$ such that

$$(1) \qquad \| X(E_n) \| \geqq \varepsilon_0, \qquad n=1, 2, \dots .$$

We may assume, without the loss of generality, that the E_n are disjoint and that we have $\alpha(\sum_{n=1}^{\infty} E_n)=\sum_{n=1}^{\infty} \alpha(E_n)<\infty$. The least Borel family $\mathfrak{E}'$ which contains all E_n evidently consists of the sets E of the form: $E=\sum_{i=1}^{\infty} E_{n_i}$, where $\{n_i\}$ $(i=1, 2, \dots)$ is an arbitrary finite or denumerable infinite subsequence of the sequeuce $\{n\}$ $(n=1, 2, \dots)$. Let $\mathfrak{Y}$ be the least closed linear manifold which contains all $X(E)$ for $E \in \mathfrak{E}'$. Then $\mathfrak{Y}$ is clearly separable. From the above inequality (1), there exists, for each n, a φ_n in $\bar{\mathfrak{Y}}$ such that $\| \varphi_n \| \leqq 1$ and

$$(2) \qquad \varphi_n X(E_n) \geqq \varepsilon_0 .$$

Now, since $\mathfrak{Y}$ is separable, there exists a subsequence $\{\varphi_{n_i}\}$ $(i=1, 2, \dots)$ of $\{\varphi_n\}$ $(n=1, 2, \dots)$ such that $\{\varphi_{n_i} X(E)\}$ $(i=1, 2, \dots)$ converges for any E in $\mathfrak{E}'$. By a theorem of Vitali-Hahn-Saks,[6] $\{\varphi_{n_i} X(E)\}$ $(i=1, 2, \dots)$ are equi-absolutely continuous, which is a contradiction to (2). Thus $X(E)$ must be strongly absolutely continuous.

3. *Proof of Theorem 2.* Since $\mathfrak{X}$ is locally weakly compact, $\bar{\mathfrak{X}}$ is also locally weakly compact.[10] Consequently, by a theorem of B. J. Pettis,[11] $\mathfrak{X}$ and $\bar{\mathfrak{X}}$ both satisfy the condition (D); i. e., any function of

6) S. Saks: Addition to the note on some functionals, Trans. Amer. Math. Soc., **35** (1933), 966-970.

The fact that the theorem of Vitali-Hahn-Saks is powerful in these problems was suggested by reading the abstract of B. J. Pettis.

7) S. Bochner-A. E. Taylof, loc. cit., Theorem 7.1 (p. 939).

8) I owe this proof to S. Kakutani.

9) B. J. Pettis: On integration in vector spaces, Trans. Amer. Math. Soc. **44** (1939), 277-304. Theorem 2.32.

10) V. Gantmakher and V. Šmulian: Sur les espaces linéaires dont la sphère unitaire est faiblement compacte, C. R. URSS, **17** (1937), 91-94. Theorem 3.

11) B. J. Pettis: Differentiation in Banach space, Duke Math. Journ., **5** (1939), 254-270. Theorem 3.1.

bounded variation with values in $\mathfrak{X}$ or $\bar{\mathfrak{X}}$ has a derivative almost everywhere. Hence in our case, by a theorem of S. Bochner-A. E. Taylor,[12] any bounded linear functional $U(f)$ defined on $L(\mathfrak{X})$ takes the form:

$$U(f)=\int_0^1 \varphi(t)f(t)\,dt,$$

where $\varphi(t)\in M(\bar{\mathfrak{X}})$[13] and $\|U\|=\operatorname{ess.}\max_{0\le t\le 1}\|\varphi(t)\|$.

Let $\{f_n(t)\}$ $(n=1,2,\ldots)$ be an equi-integrable sequence in $L(\mathfrak{X})$, and put $F_n(t)=\int_0^t f_n(s)\,ds$. $F_n(t)$ $(n=1,2,\ldots)$ are clearly strongly absolutely equi-continuous. We shall first prove that we can choose a subsequence $\{F_{n_i}(t)\}$ $(i=1,2,\ldots)$ of $\{F_n(t)\}$ $(n=1,2,\ldots)$ which converges weakly to an element $F(t)\in\mathfrak{X}$ for each t. Let $\{t_j\}$ $(j=1,2,\ldots)$ be a denumerable set which is dense in $[0,1]$. Since $\mathfrak{X}$ is locally weakly compact, we can choose a subsequence $\{F_{n_i}(t)\}$ $(i=1,2,\ldots)$ of $\{F_n(t)\}$ $(n=1,2,\ldots)$ such that $\{F_{n_i}(t_j)\}$ $(i=1,2,\ldots)$ converges weakly to an element $F(t_j)\in\mathfrak{X}$ for $j=1,2,\ldots$ Since $F_{n_i}(t)$ $(i=1,2,\ldots)$ are strongly absolutely equi-continuous, we see that $\{F_{n_i}(t)\}$ $(i=1,2,\ldots)$ converges weakly to an element $F(t)\in\mathfrak{X}$ for each t. Hence, by Vitali-Hahn-Saks' theorem, $\varphi F(t)$ is absolutely continuous for each $\varphi\in\bar{\mathfrak{X}}$. Hence, by a theorem of B. J. Pettis,[11] $F(t)$ has a derivative $f(t)\in L(\mathfrak{X})$ almost everywhere such that $F(t)=\int_0^t f(s)\,ds$ for each t.

Thus we have proved that there exists an $f(t)\in L(\mathfrak{X})$ such that

$$\lim_{i\to\infty}\int_0^t \varphi f_{n_i}(s)\,ds=\int_0^t \varphi f(s)\,ds$$

for each $\varphi\in\bar{\mathfrak{X}}$ and for each t. Consequently, by the same argument as was used by S. Bochner-A. E. Taylor,[14] we have

$$\lim_{i\to\infty}\int_0^1 \varphi(t)f_{n_i}(t)\,dt=\int_0^1 \varphi(t)f(t)\,dt$$

for each $\varphi(t)\in M(\bar{\mathfrak{X}})$. Thus the sequence $\{f_{n_i}(t)\}$ $(i=1,2,\ldots)$ converges weakly to $f(t)\in L(\mathfrak{X})$ as a sequence in $L(\mathfrak{X})$, and hereby the proof of Theorem 2 is completed.

4. *Proof of Theorem 3.* By the same argument as in the proof of Theorem 2, in case $\mathfrak{X}$ is locally weakly compact, any bounded linear functional $U(f)$ defined on $L^p(\mathfrak{X})$ $(p>1)$ takes the form:[15]

$$U(f)=\int_0^1 \varphi(t)f(t)\,dt,$$

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12) S. Bochner-A. E. Taylor, loc. cit., Theorem 3.3. (p. 921).

13) The class of all essentially bounded functions $\varphi(t)$ defined on $[0,1]$ with values in $\bar{\mathfrak{X}}$, $\|\varphi\|=\operatorname{ess.}\max_{0\le t\le 1}\|\varphi(t)\|$.

14) S. Bochner-A. E. Taylor, loc. cit., p. 923.

15) S. Bochner-A. E. Taylor, loc. cit., Theorem 3.2. (p. 920).

where $\varphi(t) \in L^q(\bar{\mathfrak{X}})$, $\frac{1}{p}+\frac{1}{q}=1$ and $\| U\|=\left(\int_0^1\| \varphi(t) \|^q dt\right)^{\frac{1}{q}}$.

Let $\{f_n(t)\}$ $(n=1, 2, \dots)$ be a sequence in $L^p(p>1)$ such that

$$\|f_n\| \equiv \left(\int_0^1 \|f_n(t)\|^p dt\right)^{\frac{1}{p}} \leqq M \qquad (M\text{: constant}),\ n=1, 2, \dots,$$

and put $F_n(t)=\int_0^t f_n(s)\,ds$. Then we can easily prove that we have

$$\text{(3)} \qquad \text{l. u. b.} \sum_{\nu=1}^{k} \| F_n(t_\nu)-F_n(t_{\nu-1})\|^p / | t_\nu - t_{\nu-1} |^{p-1} = \int_0^1 \| t_n(t) \|^p dt, \qquad n=1, 2, \dots,$$

where l. u. b. means the least upper bound for all partitions $0=t_0<t_1<\dots<t_k=1$ of the interval $[0, 1]$. Consequently, exactly as in the proof of Theorem 2, we can choose a subsequence $\{F_{n_i}(t)\}$ $(i=1, 2, \dots)$ of $\{F_n(t)\}$ $(n=1, 2, \dots)$ such that $\lim_{i\to\infty} \varphi F_{n_i}(t)=\varphi F(t)$ for each $\varphi \in \bar{\mathfrak{X}}$ and for each t, where $F(t)$ is a function with values in $\mathfrak{X}$ which clearly belongs to $V^p(\mathfrak{X})$[16] by (3). Since $\mathfrak{X}$ is locally weakly compact by assumption, by a theorem of B. J. Pettis,[11] $F(t)$ has a derivative $f(t) \in L(\mathfrak{X})$ almost everywhere such that $F(t)=\int_0^t f(s)\,ds$ for each t. Moreover, it will be easily seen that we have $f(t) \in L^p(\mathfrak{X})$.

Thus we have proved that there exist a subsequence $\{f_{n_i}(t)\}$ $(i=1, 2, \dots)$ of $\{f_n(t)\}$ $(n=1, 2, \dots)$ and an $f(t) \in L^p(\mathfrak{X})$ such that

$$\lim_{i\to\infty} \int_0^t \varphi f_{n_i}(s)\,ds = \int_0^t \varphi f(s)\,ds$$

for each $\varphi \in \bar{\mathfrak{X}}$ and for each t. Consequently we have[17]

$$\lim_{i\to\infty} \int_0^1 \varphi(t) f_{n_i}(t)\,dt = \int_0^1 \varphi(t) f(t)\,dt$$

for each $\varphi(t) \in L^q(\bar{\mathfrak{X}})$, and hereby the proof of Theorem 3 is completed.

5. *Proof of Theorem 4.* In the proof of Theorem 2 we have observed that, in case $\mathfrak{X}$ is locally weakly compact, any bounded linear functional $U(f)$ defined on $L(\mathfrak{X})$ takes the form:

$$U(f)=\int_0^1 \varphi(t) f(t)\,dt,$$

where $\varphi(t) \in M(\bar{\mathfrak{X}})$ and $\| U\| = \underset{0\leq t\leq 1}{\text{ess. max.}} \| \varphi(t) \|$.

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16) The class of all functions $f(t)$ defined on $[0, 1]$ to a Banach space $\mathfrak{X}$, such that the sums

$$\sum_{\nu=1}^{k} \|f(t_\nu)-f(t_{\nu-1})\|^p / | t_\nu - t_{\nu-1} |^{p-1}$$

are bounded for all partitions $0=t_0<t_1<\dots<t_k=1$. The least upper bound of all such sums is denoted by $V^p(f)$.

17) S. Bochner-A. E. Taylor, loc. cit., Theorem 4.1. (p. 921).

Let $\{f_n(t)\}$ $(n=1, 2, \dots)$ be a weakly convergent sequence in $L(\mathfrak{X})$. Then

$$\lim_{n\to\infty} U(f_n)=\lim_{n\to\infty}\int_0^1 \varphi(t)f_n(t)\,dt$$

exists for each $\varphi(t)\in M(\bar{\mathfrak{X}})$. As a special case, $\lim_{n\to\infty}\varphi\int_E f_n(t)\,dt$ exists for each $\varphi\in\bar{\mathfrak{X}}$ and for each measurable set E. Let us put $F_n(E)=\int_E f_n(t)\,dt$. Then, by the weak completeness of $\mathfrak{X}$, there exists a limit function $F(E)$ with values in $\mathfrak{X}$ such that $\lim_{n\to\infty}\varphi F_n(E)=\varphi F(E)$ for each $\varphi\in\bar{\mathfrak{X}}$ and for each measurable set E. By the theorem of Vitali-Hahn-Saks, the numerical functions $\varphi F_n(E)$, $n=1, 2, \dots$, are then equi-absolutely continuous for each $\varphi\in\bar{\mathfrak{X}}$. Hence $\varphi F(E)$ is completely additive and absolutely continuous for each $\varphi\in\bar{\mathfrak{X}}$. Consequently, by Theorem 1, $F(E)$ is strongly completely additive and strongly absolutely continuous. Since $\mathfrak{X}$ is locally weakly compact by assumption, by the theorem of B. J. Pettis,[11)] $F(E)$ has a derivative $f(t)\in L(\mathfrak{X})$ almost everywhere such that $F(E)=\int_E f(t)\,dt$ for each measurable set E.

Thus we have proved that there exists an $f(t)\in L(\mathfrak{X})$ such that

$$\lim_{n\to\infty}\int_E \varphi f_n(t)\,dt=\lim_{n\to\infty}\varphi\int_E f_n(t)\,dt=\int_E \varphi f(t)\,dt$$

for each $\varphi\in\bar{\mathfrak{X}}$ and for each measurable set E. Consequently, we have

$$\lim_{n\to\infty}\int_0^1 \varphi(t)f_n(t)\,dt=\int_0^1 \varphi(t)f(t)\,dt$$

for each $\varphi(t)\in M(\bar{\mathfrak{X}})$. Thus the sequence $\{f_n(t)\}$ $(n=1, 2, \dots)$ converges weakly (as a sequence in $L(\mathfrak{X})$) to $f(t)\in(\mathfrak{X})$, as was to be proved.