64. *Note on Uni-serial and Generalized Uni-serial Rings.*

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The purpose of this short note is to make some supplementary remarks on my papers "On Frobeniusean algebras" I[1] and II.[2] The remarks are about uni-serial[3] and generalized uni-serial rings[4] as well as about principal two-sided ideals.

Let A be a ring satisfying the minimum and the maximum condition for left and right ideals.[5] As an application of our study of Frobeniusean rings, we showed (F. I, Theorem 10; F. II, Theorem 16):

Theorem 1. If every two-sided ideal $\mathfrak{z}$ in A is expressible as $\mathfrak{z}=Ac=cA$ $(c\in A)$ then every residue class ring of A, including A itself, is Frobeniusean,[6] *and conversely.*

On the other hand, K. Asano proved in his paper "Verallgemeinerte Abelsche Gruppe mit hyperkomplexem Operatorenring und ihre Anwendungen":[7]

Theorem 2. If every two-sided ideal $\mathfrak{z}$ in A is expressible as $\mathfrak{z}=Ac=dA$ $(c, d\in A)$ then A is uni-serial, and conversely.[8]

Notwithstanding their apparent differences these two theorems express, as the writer realized later, one and the same fact.[9] Indeed we have, first, the following lemma, which is perhaps of some interest for itself:

Lemma 1. Let A possess a unit element. If a two-sided ideal $\mathfrak{z}$ of A is expressible as $\mathfrak{z}=Ac=dA$, then $\mathfrak{z}=cA=Ad$ too.

Proof. Denote the composition length of a left A-module $\mathfrak{m}$ by $[\mathfrak{m}]_l$. From the mapping $a\rightarrow ac$ we see readily that[10] $\mathfrak{z}=Ac$ is isomorphic to $A/l(c)=A/l(cA)$ whence $[\mathfrak{z}]_l=[A/l(cA)]_l$. But $cA\leqq\mathfrak{z}$ and

1) Ann. Math. **40** (1939) — referred to as F. I.

2) Forthcoming in Ann. Math. — referred to as F. II.

3) Einreihig. See G. Köthe, Verallgemeinerte Abelsche Gruppe mit hyperkomplexem Operatorring, Math. Zeitschr. **39** (1934). Cf. also F. I, § 7.

4) F. II, § 9. Cf. also F. I, § 2.

5) A may have an operator domain of the type described in F. II, § 4.

6) F. II, § 4. See also F. I, § 2.

7) Japanese Journ. Math. **15** (1939).

8) Not only that, every left or right ideal of a uni-serial ring is principal.
As for the connection between uni-serial rings and principal ideals cf. also the writer's note, A note on the elementary divisor theory in non-commutative domains, Bull. American Math. Soc. **44** (1938), and K. Asano, Nicht-kommutative Hauptidealringe, Act. sci. ind. (1938).

9) Incidentally, the remark adjoining the definition of generalized uni-serial rings (F. II, § 9) was rather redundant; See Lemma 1 below. But our Theorem 17 there (as well as F. I, Theorem 11) retains, of course, its original significance, since there are certainly generalized uni-serial rings which are not uni-serial. (For that Theorem 17 cf. the second half of the present note.) For instance, an algebra consisting of all matrices of a given degree $\geqq 2$ such that all the coefficients above the diagonal vanish is such.

10) We denote by $l(S)$, $S\leqq A$, the set of left annihilators of S in A.

$l(cA) \geqq l(\mathfrak{z})$. Hence

$$[\mathfrak{z}]_l \leqq [A/l(\mathfrak{z})]_l .$$

On the other hand, $Ad \cong A/l(dA) = A/l(\mathfrak{z})$ and $\mathfrak{z} \geqq Ad$. Therefore

$$[\mathfrak{z}]_l \geqq [Ad]_l = [A/l(\mathfrak{z})]_l .$$

On combining these two inequalities, we deduce $[\mathfrak{z}]_l = [Ad]_l$ whence $\mathfrak{z} = Ad$.

Similarly $\mathfrak{z} = cA$, and the lemma is proved.

Furthermore,

Lemma 2. If every residue class ring of A is Frobeniusean then A is uni-serial, and conversely.

Proof. 1) Let A satisfy our condition. On considering residue class rings with respect to powers of the radical N, we find readily that A is a generalized uni-serial ring.[1] Further, on retaining the notations in F. II (or in F. I), we consider, with a certain fixed $\varkappa$, the residue class ring $\tilde{A} = A/E_\varkappa M E_{\pi(\varkappa)}$ of A with respect to the two-sided ideal $\mathfrak{z} = E_\varkappa M E_{\pi(\varkappa)} = E_\varkappa M = M E_{\pi(\varkappa)}$ (M being the annihilator ideal of the radical; $M = l(N) = r(N)$). Now, let us assume our assertain to be true for rings of smaller composition length. $\tilde{A}$ is then uni-serial. Hence, as we find readily in analizing the structure of $\tilde{A}$

$$E_\lambda M E_\lambda \not\equiv 0 \mod \mathfrak{z}, \quad \text{or,} \quad \lambda = \pi(\lambda)$$

for *every λ different from $\varkappa$.* Therefore $\varkappa = \pi(\varkappa)$ too. It follows now that the composition residue class moduli of the left ideal AE_λ are, for each $\lambda = 1, 2, \dots$, all isomorphic to Ae_λ/Ne_λ. Hence $AE_\lambda = E_\lambda A E_\lambda$. Similarly $E_\lambda A = E_\lambda A E_\lambda$, and A is the direct sum of mutually orthogonal primary rings $E_\lambda A E_\lambda$ ($\lambda = 1, 2, \dots$). But, as A is a generalized uni-serial ring, this shows that A is uni-serial.

2) The converse is almost evident, because every residue class ring of a uni-serial ring is uni-serial again and a uni-serial ring is certainly Frobeniusean.

The equivalence of the two theorems is thus shown.

Now we turn to our second remark, which is concerned with the main theorem on gederalized uni-serial rings (F. II, Theorem 17; F. I, Theorem 11). We have observed already[2] that the converse of the theorem is true if A is an algebra. But the same is the case for general A. Namely

Theorem 3. If every directly indecomposable finite left A-module is (cyclic and is) homomorphic to a principal left ideal Ae generated by a primitive idempotent element[3] *e and if similarly every directly indecomposable finite right A-module is (cyclic and is) homorphic to a principal right ideal eA generated by a primitive idempotent element e, then A is a generalized uni-serial ring.*

To simplify our proof a little, we give first

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1) Cf. F. I, footnotes 40 and 41.

2) See the remark at the end of F. II.

3) This is equivalent to saying that the module possesses a unique maximal submodule; See F. I, § 1.

Lemma 3. Let N be the radical of A. If A/N^2 is a generalized uni-serial ring then A itself is so too. (Similarly, if A/N^2 is uni-serial then A is so too.)

Proof. Let e be an arbitrary primitive idempotent element in A. According to our assumption, not only the left module Ae/Ne but also Ne/N^2e is simple (unless it is zero). Hence N^2e is the *only* maximal left subideal of Ne. If e' is a second primitive idempotent element such that $Ne/N^2e \cong Ae'/Ne'$, then Ne is homomorphic to Ae', and by the homomorphism Ne' and N^2e' are mapped onto N^2e and N^3e respectively.[1] Thus

$$N^2e/N^3e \sim Ne'/N^2e'.$$

But the right side is, according to our assumption, simple (or zero), and therefore the left side is so too. Now we take a third idempotent element e'' such that $N^2e/N^3e \cong Ae''/Ne''$. Then we find $N^3e/N^4e \sim Ne''/N^2e''$. Continuing in this way we see that $N^ie/N^{i+1}e$ is simple for every i, unless it is zero. Thus Ae possesses a unique composition series. The same is true for eA, and thus A is a generalized uni-serial ring.

(If moreover A/N^2 is uni-serial, we may choose as $e', e'', \dots$ simply e itself. Hence A is uni-serial.)

Proof of Theorem 3. Let A satisfy our condition. In view of the above lemma, it is sufficient to treat the case where $N^2=0$. Then our purpose is to show that if e' is an arbitrary primitive idempotent element in A then the (completely reducible) left ideal Ne' as well as the (completely reducible) right ideal $e'N$ is simple, unless it is zero. Suppose therefore the contrary and assume that the left, say, ideal Ne' is neither simple nor zero. Take then two arbitrary distinct simple left subideals $\mathfrak{l}_1$ and $\mathfrak{l}_2$ in Ne'.

Case 1) Suppose first $\mathfrak{l}_1 \not\cong \mathfrak{l}_2$. We take primitive idempotent elements e_1 and e_2 such that $\mathfrak{l}_1 \cong Ae_1/Ne_1$ and $\mathfrak{l}_2 \cong Ae_2/Ne_2$, and consider the *right ideals*

$$\mathfrak{r}_1 = e_1A \quad \text{and} \quad \mathfrak{r}_2 = e_2A.$$

Since $\mathfrak{l}_1 \cong Ae_1/Ne_1$, we have $e_1Ne' \neq 0$. Hence the completely reducible right ideal e_1N ($\subseteqq \mathfrak{r}_1$) possesses a simple right subideal $\mathfrak{s}_1$ isomorphic to $e'A/e'N$. Similarly e_2N ($\subseteqq \mathfrak{r}_2$) possesses a simple right subideal $\mathfrak{s}_2$ isomorphic to $e'A/e'N$ (which is $\cong \mathfrak{s}_1$). Now, we identify the subideals $\mathfrak{s}_1$ and $\mathfrak{s}_2$ of $\mathfrak{r}_1$, $\mathfrak{r}_2$ and thus construct a new *right module* $\mathfrak{n}$:

$$\mathfrak{n} = (\mathfrak{r}_1, \mathfrak{r}), \qquad \mathfrak{r}_1 \frown \mathfrak{r}_2 = \mathfrak{s}_1 = \mathfrak{s}_2.$$

This module $\mathfrak{n}$ is directly indecomposable. For, suppose the contrary and let

$$\mathfrak{n} = \mathfrak{t}_1 + \mathfrak{t}_2 \qquad (\mathfrak{t}_1, \mathfrak{t}_2 \neq 0).$$

Then $\quad \mathfrak{n}/\mathfrak{n}N = (\mathfrak{r}_1, \mathfrak{n}N)/\mathfrak{n}N + (\mathfrak{r}_2, \mathfrak{n}N)/\mathfrak{n}N = (\mathfrak{t}_1, \mathfrak{n}N)/\mathfrak{n}N + (\mathfrak{t}_2, \mathfrak{n}N)/\mathfrak{n}N.$

But here the two decompositions must be identical, since the factors are not isomorphic to each other. Let, for instance,

1) For all this cf. F. I, § 1.

$$(\mathfrak{r}_1, \mathfrak{n}N)/\mathfrak{n}N=(\mathfrak{t}_1, \mathfrak{n}N)/\mathfrak{n}N, \qquad (\mathfrak{r}_2, \mathfrak{n}N)/\mathfrak{n}N=(\mathfrak{t}_2, \mathfrak{n}N)/\mathfrak{n}N.$$

Then there exists an element $t_1 \in \mathfrak{t}_1 e_1$ which is not contained in $\mathfrak{n}N$ (whence *a fortiori* not contained in $\mathfrak{t}_1 N$). $t_1 e_1 A = t_1 A = \mathfrak{t}_1$, because $\mathfrak{t}_1/\mathfrak{t}_1 N$ $\left(\cong (\mathfrak{t}_1, \mathfrak{n}N)/\mathfrak{n}N\right)$ is simple whence $\mathfrak{t}_1 N$ is the only maximal submodule of $\mathfrak{t}_1$. Now we want to show that $t_1 e_1 A = \mathfrak{t}_1$ is isomorphic to $\mathfrak{r}_1 = e_1 A$. For that purpose, put $t_1 = u + v$ ($u \in \mathfrak{r}_1$, $v \in \mathfrak{r}_2$); here u and v are not uniquely determined, but are unique mod $\mathfrak{s}_1 = \mathfrak{s}_2$. It follows, since $(\mathfrak{t}_1, \mathfrak{n}N) = (\mathfrak{r}_1, \mathfrak{n}N)$, that $u \notin \mathfrak{r}_1 N$ but $v \in \mathfrak{r}_2 N$. Thus $ue_1 \notin \mathfrak{r}_1 N$ either and $ue_1 A = \mathfrak{r}_1$. $ua \leftrightarrow a$ ($a \in e_1 A = \mathfrak{r}_1$) is an (operator-) automorphism of $\mathfrak{r}_1$. Now, suppose $t_1 a = 0$ for an element a in $e_1 A$. Then $ua + va = 0$ whence

$$ua = -va \in \mathfrak{s}_1 = \mathfrak{s}_2,$$

and therefore $a \in e_1 N$, since $\mathfrak{s}_1 \leqq e_1 N$. But then va $(\in \mathfrak{r}_2 N e_1 N) = 0$, whence $ua = 0$ and $a = 0$. It follows now that the homomorphic mapping $a \rightarrow t_1 a$ ($a \in e_1 A$) is indeed an isomorphism between $t_1 A = \mathfrak{t}$ and $\mathfrak{r}_1 = e_1 A$.

Similarly $\mathfrak{t}_2 \cong \mathfrak{r}_2$. But this is a contradiction, as a simple computation of composition lengths shows, and thus $\mathfrak{n}$ must be directly indecomposable.

On the other hand, $\mathfrak{n}$ can not be homomorphic to a principal right ideal generated by a primitive idempotent element, because $\mathfrak{n}/\mathfrak{n}N$ is not simple. Thus we are led to a contradiction, and therefore the case 1) can not occur.

Case 2) $\mathfrak{l}_1 \cong \mathfrak{l}_2$. On putting $\mathfrak{l} = Ae'$, we consider a second left module $\mathfrak{m}$ which is isomorphic to $\mathfrak{l}$. Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be simple submoduli of $\mathfrak{m}$ corresponding respectively to $\mathfrak{l}_1$ and $\mathfrak{l}_2$ by the isomorphism. We then identify $\mathfrak{l}_2$ with $\mathfrak{m}_1$, to obtain a new *left module* $\mathfrak{n}$:

$$\mathfrak{n} = (\mathfrak{l}, \mathfrak{m}), \qquad \mathfrak{l} \frown \mathfrak{m} = \mathfrak{l}_2 = \mathfrak{m}_1.$$

Then we find without much difficulty that $\mathfrak{n}$ is directly indecomposable; for the detail of the proof cf. G. Köthe, l. c., p. 40 or § 5 of H. Brummund's thesis.[1] But this left module $\mathfrak{n}$ is not homomorphic to a principal left ideal generated by a primitive idempotent element, for $\mathfrak{n}/N\mathfrak{n}$ is not simple. Thus Case 2) is also impossible.

We deduce now that Ne' must be simple (or zero). Similarly $e'N$ is always simple (or zero). Hence A is a generalized uni-serial ring, q. e. d.

In connection with his theory of uni-serial rings, G. Köthe propounded the problem to determine the general type of rings A (possessing a unit element and satisfying the minimum condition (whence also the maximum condition) for left and right ideals) such that every A-module is a direct sum of cyclic submoduli.[2] For commutative rings the notion of uni-serial rings settles this problem. As for non-commutative rings, this is not the case. In fact, generalized uni-serial rings enjoy the proposed property. But, even generalized uni-serial rings are

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1) H. Brummud, Über Gruppenringe mit einem Koeffizientenkörper der Charakteristik p, Dissertation Münster (1939).

2) G. Köthe, l. c., § 2.

not general enough, and the writer regrets to have to leave the question open here. For instance, let A be an algebra consisting of all matrices (in a given field) of the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ c & d & a \end{pmatrix}.$$

Then every right A-module is, as a rather complicated group-theoretical consideration shows, a direct sum of cyclic submoduli each of which is homomorphic to a principal right ideal generated by a primitive idempotent element, and moreover every left A-module is a direct sum of cyclic submoduli which are however not necessarily homomorphic to a principal left ideal generated by a primitive idempotent element.[1] But A is not a generalized uni-serial ring.

Further, H. Brummund showed in his paper l. c. that a non-cyclic p-group always possesses arbitrary large directly indecomposable representations in a field of characteristic p, that is, the group algebra of a non-cyclic p-group over a field of characteristic p has arbitrary large directly indecomposable left, say, moduli. Indeed, his argument shows that *the same holds for A such that a left module $N^{i-1}e/N^{i}e$, where N is the radical and e is a suitable primitive idempotent element, contains at least two simple submoduli isomorphic to each other.* Now arises the problem to determine general type of rings which possess arbitrary large directly indecomposable left or right moduli. But, the writer has to leave also this problem open; the notion of generalized uni-serial rings is, *a fortiori*, too special to settle this question.

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1) Thus this example shows also that in the above theorem 2 it is essential to consider both left and right moduli at the same time.