81. On the General Zetafuchsian Functions.[1)]

By Masao SUGAWARA.
Tokyo Bunrikwa-Daigaku, Tokyo.

(Comm. by T. TAKAGI, M.I.A., Oct. 12, 1940.)

1. Introduction.

In a previous paper[2)] I have constructed a theory of automorphic functions of higher dimensions. Using the same notations as in that paper, we will call the space $\mathfrak{A}$ the set of symmetrical matrices Z of the dimension n with the condition $E > Z'\bar{Z}$. We put now $R(Z) = -\log|E-\bar{Z}Z|$, that is $|E-\bar{Z}Z| = e^{-R(Z)}$. Then $R(Z)\to o$ or ∞ according as $Z\to O$ or Z tends to the boundary of the space $\mathfrak{A}$ and conversely.

Matrices U of dimension $2n$ satisfying the conditions $U'JU=J$, $U'S\bar{U}=S$, where $J=\begin{pmatrix} O & E \\ -E & O\end{pmatrix}$ and $S=\begin{pmatrix} E & O \\ O & -E\end{pmatrix}$ are of the form $U=\begin{pmatrix} U_1 & U_2 \\ U_3 & U_4\end{pmatrix}=\begin{pmatrix} \bar{U}_4 & \bar{U}_3 \\ U_3 & U_4\end{pmatrix}$ and form a group Γ.

When Z is an inner or a boundary point of $\mathfrak{A}$, $W=(U_1Z+U_2)(U_3Z+U_4)^{-1}$ is also an inner or a boundary point of $\mathfrak{A}$ respectively, so that we called this transformation a displacement of the space $\mathfrak{A}$ induced by $U\in\Gamma$. We then have

Lemma 1. If $W=(U_1Z+U_2)(U_3Z+U_4)^{-1}$, $U=\begin{pmatrix} U_1 & U_2 \\ U_3 & U_4\end{pmatrix}\in\Gamma$, then

$$\|U_3Z+U_4\|^{-2}=\frac{|E-\bar{W}W|}{|E-\bar{Z}Z|}, \text{ that is } \|U_3Z+U_4\|^{-2}=e^{R(Z)-R(W)}.$$

Proof. Every point of $\mathfrak{A}$ can be represented in the form $Z=PQ^{-1}$, where P, Q make a pair of symmetrical matrices with the condition $|Q|\neq 0$. Let $\begin{pmatrix} P_1 \\ Q_1\end{pmatrix}=U\begin{pmatrix} P \\ Q\end{pmatrix}$ then P_1, Q_1 make also a pair of symmetrical matrices with the condition $|Q_1|\neq 0$, such that $W=P_1Q_1^{-1}$, and

$$\begin{aligned} E-\bar{W}W &= E-\bar{Q}_1'^{-1}\bar{P}_1'P_1Q_1^{-1}=\bar{Q}_1'^{-1}(\bar{Q}_1'Q_1-\bar{P}_1'P_1)Q_1^{-1} \\ &= \overline{(U_3P+U_4Q)}'^{-1}(\bar{Q}'Q-\bar{P}'P)(U_3P+U_4Q)^{-1} \\ &= \overline{(U_3Z+U_4)}'^{-1}(E-\bar{Z}Z)(U_3Z+U_4)^{-1}\end{aligned}$$

Taking the determinants, the lemma follows at once.

---

1) Cf. H. Poincaré. Memoire sur les fonctions fuchsiennes. Memoire sur les fonctions zetafuchsinnes (Oeuvre Tom II.)

2) Masao Sugawara. Über eine allgemeine Theorie der fuchssche Gruppen und Theta-Reihen. Ann. Math. **41**; cited with S.

2. Riemannian space.

Now we introduce a noneuclidean metric in the space $\mathfrak{A}$, namely we take as a line-element an expression $ds=\sqrt{Sp\,dA(E-\bar{A}A)^{-1}d\bar{A}'(E-A\bar{A})^{-1}}$, invariant under $\mathfrak{B}$[1] and consider $\mathfrak{A}$ as a riemannian space with this metric.

Any subgroup $\mathfrak{G}$ of the group $\mathfrak{B}$ having no infinitesimal element is called a fuchsian group. As in the classical case for one variable[2] the space $\mathfrak{A}$ can be divided into "normal polyhedrons". i. e. congruent convex polyhedrons such that any two inner points of each normal polyhedron are not equivalent by the displacements of the group $\mathfrak{G}$, while the sides are equivalent in pairs.

A normal polyhedron can be taken as a fundamental domain of the group $\mathfrak{G}$. The number of normal polyhedrons having an inner point of $\mathfrak{A}$ in common is finite.

Lemma 2. If the noneuclidean distance of a point B from an inner point A is sufficiently small, then $|R(A)-R(B)|$ is uniformly bounded with respect to A.

Proof. Let H be the image of B by the displacement

$$U_A=\begin{pmatrix} C_1 & O \\ O & \bar{C}_1 \end{pmatrix}\begin{pmatrix} E & -A \\ -\bar{A} & E \end{pmatrix}=\begin{pmatrix} C_1 & -C_1A \\ -\bar{C}_1\bar{A} & \bar{C}_1 \end{pmatrix},\ C_1=(C')^{-1},\ E-A'\bar{A}=C'\bar{C},$$

which transforms A into O. Put

$$U_A^{-1}=\begin{pmatrix} \bar{C}_1' & AC_1' \\ \bar{A}\bar{C}_1' & C_1' \end{pmatrix}=\begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}, \qquad \text{then}$$

$$|R(A)-R(B)|=\log\frac{|E-\bar{B}B|}{|E-\bar{A}A|}=\log\frac{|E-\bar{H}H|}{|E|}+\log\frac{||U_4||^2}{||U_3H+U_4||^2}$$

$$=\log|E-\bar{H}H|-\log||U_4^{-1}U_3H+E||^2.$$

Here $U_4^{-1}U_3=C_1'^{-1}\bar{A}\bar{C}_1'$ is bounded, because

$$Sp(C_1'^{-1}\bar{A}\bar{C}_1')\overline{(C_1'^{-1}\bar{A}\bar{C}_1')'}=Sp(C\bar{A}\bar{C}^{-1})(C'^{-1}A\bar{C}')=Sp\bar{C}'C\bar{A}\bar{C}^{-1}C'^{-1}A$$

$$=Sp(E-\bar{A}A)\bar{A}(E-A\bar{A})^{-1}A=Sp\bar{A}(E-A\bar{A})(E-A\bar{A})^{-1}A=Sp\bar{A}A\leqq n.$$

On the other hand, when B is near to A, H is also near to O. Hence $|R(A)-R(B)|$ is bounded.

The volume V in the riemannian space $\mathfrak{A}$ is given by

$$V=2^{\frac{n(n-1)}{2}}\int\frac{dZ^*}{|E-\bar{Z}Z|^{n+1}},$$

where $$dZ^*=\prod_{i\leq j}da_{ij}db_{ij},\ Z=(a_{ij}+b_{ij}i),$$

1) S. p. 491. $\mathfrak{B}$ means the group of all displacements of $\mathfrak{A}$.

2) H. Weyl. Die Idee der Riemannschen Fläche, § 20.

because the determinant of the quadratic form $ds^2 = SpdPd\bar{P}'$ in a_{ij}, b_{ij}, where $dP = C'^{-1}dZC^{-1}$, is $= 2^{n(n-1)} |C'^{-1}C^{-1}|^{n+1} |\bar{C}'^{-1}\bar{C}^{-1}|^{n+1} = 2^{n(n-1)} |E - \bar{Z}Z|^{-2(n+1)}$, as we can verify by calculation.

By a suitable unitary transformation U, the non-negative herimitian form $Z\bar{Z}$ can be transformed into the diagonal form $D = \begin{pmatrix} \alpha_1^2 \cdots\cdots 0 \\ \cdots\cdots\cdots \\ 0 \cdots\cdots \alpha_n^2 \end{pmatrix}$, $0 \leqq \alpha_i^2 \leqq 1$. Let $UZ = Y$, then $Y\bar{Y}' = D$ and the elements y_{ik} of Y are not greater than 1 in absolute values. Hence we obtain an evalution for the value of the domain $R(Z) \leqq R(B)$, B being a given point,

$$V = \int_{R(Z) \leq R(B)} \frac{dY}{|E - \bar{Z}Z|^{n+1}} \leqq \frac{2^{\frac{n(n-1)}{2}}}{|E - \bar{B}B|^{n+1}} \int dY$$

$$\leqq \frac{2^{n^2} \pi^{\frac{n(n+1)}{2}}}{|E - \bar{B}B|^{n+1}} = 2^{n^2} \pi^{\frac{n(n+1)}{2}} e^{(n+1)R(B)}.$$

Lemma 3. Let Z be an inner point of a normal polyhedron F of a fuchsian group $\mathfrak{G}$. Let σ be the volume of a (noneuclidean) sphere k around the point Z with so small a radius that it lies entirely in F. Then the number $N(B)$ of equivalent points to Z with respect to the group $\mathfrak{G}$ in the domain $R(A) \leqq R(B)$, B being a given point, is at most

$$2^{n^2} \pi^{\frac{n(n+1)}{2}} \sigma^{-1} e^{(n+1)(R(B)+\lambda)}.$$

Here λ means a finite constant depending on the radius of the sphere k but not on B and Z.

Proof. Let λ be the quantity such that from (the (noneuclidean) distance of Q from P) $\leqq$ (the radius of k), follows $|R(P) - R(Q)| \leqq \lambda$. λ exists and is independent of P by the lemma 2.

Then the total volume of the images of the sphere k whose centres lie in the domain $R(A) \leqq R(B)$ does not exceed the volume of the domain $R(A) \leqq R(B) + \lambda$, namely $N(B)\sigma \leqq 2^{n^2} \pi^{\frac{n(n+1)}{2}} e^{(n+1)(R(B)+\lambda)}$.

The same inequality also holds when Z lies on the boundary of F, but not on the boundary of $\mathfrak{A}$, and the radius of the sphere k is sufficiently small, if we multiply an absolute constant g to the right.

3. Thetafuchsian function.

From the previous consideration, we can easily deduce the absolute and uniform convergence of a thetafuchsian function

$$\theta_k(Z) = \sum_{U \in \mathfrak{G}} \frac{1}{|U_3 Z + U_4|^{2k(n+1)}}, \quad k > 1,$$

where $\sum$ sums over all the displacements $U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$ of a fuchsian group $\mathfrak{G}$.

Indeed, denote with $\mathfrak{k}_m$ the domain $R(A) \leqq mr, r > 0$ and put $\mathfrak{d}_m = \mathfrak{k}_m - \mathfrak{k}_{m-1}$. Divide the terms of the series $\sum$, whose terms are the absolute values of the terms of $\theta_k(Z)$, into parts $G_1, G_2, \cdots$ such that G_m is the sum of terms corresponding to displacements U for which $W = (U_1Z+U_2)(U_3Z+U_4)^{-1}$ lie in the domain $\mathfrak{d}_m$. Then every term of G_m is smaller than $e^{(n+1)k(R(Z)-(m-1)r)}$ and the number of such terms is at most $2^{n^2}\pi^{\frac{n(n+1)}{2}} g\sigma^{-1}e^{(n+1)(mr+\lambda)}$, so that

$$G_m < 2^{n^2}\pi^{\frac{n(n+1)}{2}} g\sigma^{-1}e^{(n+1)(\lambda+mr)}e^{(n+1)k(R(Z)-(m-1)r)} = C_0e^{-(n+1)m(k-1)R(Z)},$$

where C_0 is a constant independent of m. Hence follows the absolute and uniform convergence of $\theta_k(Z)$ in $\mathfrak{A}$ when $k > 1$.

4. Zetafuchsian functions.

Let $\mathfrak{G}$ be a fuchsian group with finite number of generators. Let Γ be a representation by matrices of the dimension p of the group $\mathfrak{G}$. Let $\sum_1, \sum_2, \cdots, \sum_\nu$ be the matrices corresponding to generators of $\mathfrak{G}$ and their inverses, so that any substitution S of Γ can be written in the form $S = \sum_1^{a_1}\sum_2^{a_2}\cdots\sum_q^{a_q}$, where $a_1, a_2, \cdots, a_q > 0$; we call the minimum ρ of the number $a_1+a_2+\cdots+a_q$ the exponent of S. The exponent of S is not greater than that of the corresponding element S_0 of $\mathfrak{G}$.

Theorem: The series

$$Z_k(Z) = \sum_{S_0 \in \mathfrak{G}} S^{-1}\,|\,U_3Z+U_4\,|^{-2k(n+1)}$$

is absolutely and uniformly convergent in $\mathfrak{A}$ when k is sufficiently large. Hereby $S_0 = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$ and S is the element of Γ corresponding to S_0. (The terms and the value of $Z_k(Z)$ is thus p dimensional matrices.)

For a displacement $T_0 = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ of $\mathfrak{G}$ holds the transformation formula,

$$Z_k\big(T_0(Z)\big) = |\,T_3Z+T_4\,|^{2k(n+1)}TZ_k(Z),$$

where T is the element of Γ corresponding to T_0.

We will call the series $Z_k(Z)$ a zetafuchsian function of Z

Proof. The second part is evident. For the proof of the 1st part, we can assume without loss of generarity that 0 and Z lie in the same normal polyhedron R_0.

Join 0 with S_0Z by an arc l of a geodesic in $\mathfrak{A}$. Let $R_0, R_1, \cdots, R_t$ be the normal polyhedrons through which l passes successively. Then we have clearly $\rho \leqq t$, where ρ is the exponent of S. Let γ be the greatest lower bound of the length of curves joining in R_0 two sides of R_0 without a common point, then $\gamma > 0$. Let C_0 be a common side of R_0 and R_1, C_1 that of R_1 and R_2 and C_2 a side of R_3 such that C_0, C_1, C_2 have no point in common. Let μ_0 the greatest lower bound of the length of curves connecting C_0 and C_2 and passing through C_1; $\mu_1, \mu_2, \cdots, \mu_{t-2}$ be the corresponding numbers with respect to $R_1R_2R_3, R_2R_3R_4, \cdots, R_{t-2}R_{t-1}R_t$. Let μ be the smallest of $\mu_0, \mu_1, \cdots, \mu_{t-2}$, then $\mu > 0$. The number of normal polyhedrons having a point in common is finite. Let h be the least upper bound of the number of such polyhedrons. h is obviously also finite. Then we get $t \leqq R^*\left(\frac{1}{\gamma}+\frac{h}{\mu}\right)$, where R^* means the shortest distance from 0 to S_0Z. Consequently we have $t < aR^*$ with a constant a; hence $\rho < aR^*$. Now the noneuclidean length l_A of the euclidean straight line from 0 to $A=S_0Z$ is

$$l_A=\int_0^1 d\lambda \sqrt{SpA(E-\lambda^2\bar{A}A)^{-1}\bar{A}(E-\lambda^2A\bar{A})^{-1}}$$
$$=\int_0^1 d\lambda \sqrt{Sp(E-\lambda^2A\bar{A})^{-1}A\bar{A}(E-\lambda^2A\bar{A})^{-1}}\ . \qquad \begin{pmatrix}\text{to be integrated}\\ \text{along the real}\\ \text{value of } \lambda.\end{pmatrix}$$

From the relations

$$A(E-\lambda^2\bar{A}A)=(E-\lambda^2A\bar{A})A\,, \qquad A(E-\lambda^2\bar{A}A)^{-1}=(E-\lambda^2A\bar{A})^{-1}A\,,$$

and

$$Sp(E-\lambda^2A\bar{A})^{-1}A\bar{A}(E-\lambda^2A\bar{A})^{-1}$$
$$=Sp\{\bar{U}'(E-\lambda^2D)U\}^{-1}\bar{U}'DU\{\bar{U}'(E-\lambda^2D)U\}^{-1}$$
$$=SpU^{-1}(E-\lambda^2D)^{-1}U\bar{U}'DUU^{-1}(E-\lambda^2D)^{-1}U$$
$$=Sp(E-\lambda^2D)^{-1}D(E-\lambda^2D)^{-1}=\sum_{i=1}^n\left(\frac{a_i}{1-\lambda^2a_i^2}\right)^2,$$

where U is a unitary matrix such that $UA\bar{A}\bar{U}'=D=\begin{pmatrix}a_1^2\cdots\cdots 0\\ \cdots\cdots\cdots\\ 0\cdots\cdots a_n^2\end{pmatrix}$ is of a diagonal form, it follows that

$$l_A=\int_0^1\sqrt{\sum_{i=1}^n\left(\frac{a_i}{1-\lambda^2a_i^2}\right)^2}\,d\lambda<\sum_{i=1}^n\int_0^1\frac{a_i}{1-\lambda^2a_i^2}\,d\lambda$$
$$=\frac{1}{2}\log\prod_{i=1}^n\frac{1+a_i}{1-a_i}\,.$$

Hence we have

$$2R^*<2l_A<\log\prod_{i=1}^n(1+a_i)^2-\log|E-\bar{A}A|\,, \quad R^*<n\log 2+\frac{1}{2}R(A)\,,$$

therefore $\rho<C+\frac{a}{2}R(A)$, where C and a means constants. On the other hand, let M be greatest among the absoltue values of the element of $\sum_1, \sum_2, \cdots, \sum_\nu$. Then the elements S are not greater than $(pM)^\rho < C_0e^{\frac{a}{2}R(A)\log Mp}$ in absolute value, where C_0 means a constant.

Hence we get

$$(Mp)^{\rho}\,\|\,U_3Z+U_4\,\|^{-2(n+1)k}<C_0e^{-(n+1)k(R(A)-R(Z))+\frac{a}{2}R(A)\log Mp},$$

from the relation

$$\|\,U_3Z+U_4\,\|^{-2(n+1)k}=e^{-(n+1)k(R(A)-R(Z))}.$$

Now let us take k so large that $(n+1)k-2(n+1)>\frac{a}{2}\log Mp$. Let $s_{\mu\nu}$ be an element of S^{-1}. Then we have

$$|\,s_{\mu\nu}\,|\,\|\,U_3Z+U_4\,\|^{-2k(n+1)}<\|\,U_3Z+U_4\,\|^{-4(n+1)}.$$ [1]

Therefore the series $Z_k(Z)$ is absolutely and uniformly convergent in $\mathfrak{A}$.

I express my hearty thanks to Mr. S. Iyanaga for his kind criticism.

1) Because the exponent of S^{-1} is equal to that of S.