115. Lattice Theoretic Foundation of Circle Geometry.

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By G. Birkhoff and K. Menger the lattice theoretic foundation of projective geometry, affine geometry and non-euclidean geometry was established, from which continuous geometry of von Neumann was obtained as a generalization. On the other hand, the axiomatic foundation of the circle geometry was already established[1], but axioms of projective, affine and non-euclidean geometries are not sufficient for circle geometry. The object of this paper is therefore to give the system of axioms of circle geometry from the lattice theoretic standpoint.

1. Let L be a system of elements $a, b, c, \dots$ which satisfies the following axioms.

Axiom 1. L is a partially ordered system with 0 and I, that is, the relation "$\leqq$" is defined and

1.1. $a \leqq a$.

1.2. $a \leqq b$ *and* $b \leqq c$ *imply* $a \leqq c$.

1.3. *there exist* 0 *and* I *such that* $0 \leqq a \leqq I$ *for all* a *in* L.

Axiom 2. L is a semi-lattice, that is, the relations join "$\cup$" and meet "$\cap$" are defined and

2.1. *for every* a *and* b *in* L, *there exists* $c = a \cup b$, *such that* (*i*) $a \leqq c$, $b \leqq c$ *and* (*ii*) $a \leqq d$, $b \leqq d$ *imply* $c \leqq d$.

2.2. *if* $e = a \cap b$ *exists, then* (*i*) $e \leqq a$, $e \leqq b$ *and* (*ii*) $f \leqq a$, $f \leqq b$ *imply* $f \leqq e$.

Axiom 3. L is finite dimensional, that is, for every a in L there corresponds a finite positive integer $d(a)$, called dimension function such that

3.1. $d(0) = -1$, $d(I) > 3$.

3.2. $1 + max\big(d(a), d(b)\big) \leqq d(a \cup b) \leqq 1 + d(a) + d(b)$ *if* $a \cup b \neq a, b$.

Axiom 4. L is modular in a restricted sense, that is,

4.1. *if* (*i*) $a \leqq c$, (*ii*) $(a \cup b) \cap c$ *and* $b \cap c$ *exist and* (*iii*) c *covers* a *or* b *covers* $b \cap c$, *then*

$$(a \cup b) \cap c = a \cup (b \cap c).$$

2. We will show that circle geometry satisfies these axioms. Let us consider a system $\mathfrak{L}$ of elements of the following kinds: point, a pair of points, circle, sphere, ..., whose dimensions are $0, 1, 2, 3, 4, \dots$, respectively.

Taking $\leqq$ as incidence relation, $\mathfrak{L}$ satisfies Axiom 1. Concerning Axiom 2, join $\cup$ is interpreted as follows: join of two points is a pair of points, join of a point and a pair of points is a circle determined

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1) This point has been communicated by Prof. T. Kubota. This problem was suggested to the author by Mr. S. Kyo whom he expresses his hearty thanks.

by these three points, join of a point and a circle is a sphere generated by circles determined by three points which are generated by a given circle, and so on. If two elements have common points, then the meet of the two elements is interpreted as the common part of them. Some non-intersecting elements *do not* have meets and some have meet 0. This will be illustrated later.

Axiom 3 is easily verified. If a and b are pairs of points on a circle c, then $c=a \cup b$ and

$$d(a \cup b)=1+\max\big(d(a),\, d(b)\big)\,.$$

If a is a point and b is a pair of points, then

$$d(a \cup b)=1+d(a)+d(b)\,.$$

Verification of Axiom 4 is easy.

Axiom 1–4 are more general than those of projective and affine geometries. Therefore these geometries enter under these axioms. Modifying Axiom 4 in the following form:

Axiom 4 is not true when condition (iii) is taken off, we can distinguish circle geometry from projective and affine geometries.

3. We will further add the following axiom:

Axiom 5. If an element covering 0 *is represented by* p, *then*

5.1. *not* $p \leqq a$ *implies the existence of* $p \cap a$ *which is* 0.

5.2. $p \leqq a$ *and* $p \leqq b$ *imply the existence of* $a \cap b$.

Then we can develop the incidence geometry of Whittaker and Baker as Gorn[1] did.

4. *Axiom 6. L is complemented, that is, for any a in L, there is an element a′ called complement of a such that*

6.1. $a \cup a'=I$.

6.2. $d(a)+d(a')+1=d(I)$.

Definition 1. If a_1 *and* a_2 *have a common complement, then* a_1 *and* a_2 *are said to be in perspective and* $a_1 \sim a_2$ *in symbol.*

Thus we can develop the theory of perspectivity.

5. For the sake of simplicity we consider the 4-dimensional case—in model, 3-dimensional circle geometry. We call the two-dimensional element as circle.

We devide all circles in L into classes in the following way: for every point a, a class contains all circles containing a such that among them there is no triple of circles intersecting only on a.

Definition 2. If two circles c_1 *and* c_2 *belong to the same class, then they are said to be congruent and write* $c_1 \equiv c_2$ *symbolically.*

Definition 3. For a pair of circles c_1 *and* c_2, *if there are exactly two classes* $\mathfrak{R}_1$ *and* $\mathfrak{R}_2$ *such that (i) each class contains circles which intersects* c_1 *and* c_2 *at a point respectively and (ii) such circles belong to one of* $\mathfrak{R}_1$ *and* $\mathfrak{R}_2$ *but not to the other, then* c_1 *and* c_2 *are said to be parallel and write* $c_1 \| c_2$ *symbolically.*

If $c_1 \| c_2$, then we define $c_1 \cap c_2=0$. If c_1 and c_2 are not parallel and the hypothesis of 5.2 is not satisfied, then $c_1 \cap c_2$ does not exist.

1) S. Gorn, Bull. Am. Math. Soc., 1940.

Thus we can develop the theory of parallelism.

We must remark that it may happen that all elements of the same dimension are congruent.

6. We will conclude this paper by the following remark. Considering the system of elements of the following kinds: point, a pair of points, a triple of points, parabola, paraboloid, ...; or point, a pair of points, a triple of points, a quadruple of points, (non-parabolic) conic, conicoid, ...; etc., we know that "parabola" geometry, "conic" geometry, etc. satisfy our axioms. It is easy to distinguish these geometries by axioms.

For example,

Axiom 7[1]. If a and b are given such that

$$a \frown b=0\,, \quad d(a)=1\,, \quad d(b)=2\,,$$

then there are exactly two elements c and d such that

$$a \subset c\,, \quad a \subset d\,, \quad d(c)=d(c)=d(d)=2\,, \quad d(c \frown b)=d(d \frown b)=0\,.$$

This distinguishes also circle geometry from affine geometry.

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1) This axiom is due to Mr. S. Kyo.