PAPERS COMMUNICATED

13. *On Regularly Convex Sets.*

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§ 1. *Introduction.* We denote by E a Banach space and by $\bar{E}$ its conjugate space. A set $X \leqq E$ is called *convex* if $x, y \in E$ implies $ax+(1-a)y \in E$ for all $a\,(1 \geqq a \geqq 0)$. According to M. Krein and V. Smulian[1], a set $F \leqq \bar{E}$ is called *regularly convex* if for every $g \bar{\in} F$ $(g \in \bar{E})$ there exists $x_0 \in E$ such that $\sup_{f \in F} f(x_0) < g(x_0)$. It is easy to see that only convex sets in $\bar{E}$ may be regularly convex. Moreover we may prove

Theorem 1. A convex set $F \leqq \bar{E}$ is regularly convex if and only if F is closed in the weak topology of $\bar{E}$ as functionals.

Hereby, for any $f_0 \in \bar{E}$, its weak neighbourhood $U(f_0, x_1, x_2, \dots, x_n, \varepsilon)$ is defined as the totality of $f \in \bar{E}$ such that $\sup_{1 \leq i \leq n} |f(x_i) - f_0(x_i)| < \varepsilon$, where $\{x_i\}_{i=1,2,\dots,n}$ is an arbitrary system of points $\in E$ and ε is an arbitrary positive number[2].

The purpose of the present note is to show that there exists a kind of duality between (strongly) closed convex sets $\leqq E$ and regularly convex sets $\leqq \bar{E}$. By this duality we may give, to almost all the theorems in Chapter 1 of $[K-S]$, geometrical interpretations and new proofs. We then give a proof to Krein-Milman's[3]

Theorem 2. If a (strongly) bounded set $F \leqq \bar{E}$ is regularly convex, then F has extreme points f_0, viz. points f_0 such that $f_0 \neq \frac{1}{2}(g+h)$ for any two $g, h \in F$, $g \neq f_0$, $h \neq f_0$.

If E is separable, the above duality shows that Theorem 2 is an immediate corollary of a theorem due to S. Mazur[4]. The proof for non-separable E is also given by tranfinite induction. This was obtained by one of us (Fukamiya): Zenkoku Shijyô-Sugaku Danwakai, 207 (Japanese). After the present note is completed, we received a letter from S. Kakutani, now in Princeton, and we knew that F. Bohnenblust

1) Ann. of Math., **41** (1940), 556–583, to be referred as $[K-S]$.

2) The importance of weak topologies in the theory of Banach spaces was especially stressed on by S. Kakutani with much success: Proc. **15** (1939), 169–173 and **16** (1940), 63–67. We omit the easy proof of Theorem 1, since it is similar to his proof of the equivalence between the transfinite closure and the regular closure of linear subspaces $\leqq \bar{E}$. The equivalence of the two notions: transfinitely closed convex sets $\leqq \bar{E}$ and regularly convex sets $\leqq \bar{E}$, was proved in $[K-S]$, 569.

3) The vol. 9 of the Stud. Math. is not yet arrived at our institute. We knew their result from M. and S. Krein's paper: C. R. URSS, **27**, 5 (1940), 427–430.

4) Stud. Math., **4** (1933), 70–84.

also obtained the same proof. Kakutani also says in the letter that Chap. 1 of [K−S] may be rewritten by his method (Theorem 1).

§ 2. *The duality.* For any $X \leqq E$, let X^* denote the totality of $f \in \bar{E}$ such that $\sup_{x \in X} f(x) \leqq 1$. In the same way, for any set $F \leqq \bar{E}$, let F' denote the totality of $x \in E$ such that $\sup_{f \in F} f(x) \leqq 1$. Then we have[1)]

Lemma 1. $X^{*\prime *} = X^*$, $F^{\prime * \prime} = F'$.

Proof. Surely $X \leqq X^{*\prime}$ and $F \leqq F^{\prime *}$. Moreover we have $X_1^* \leqq X_2^*$ $(F_1' \leqq F_2')$ from $X_1 \geqq X_2$ $(F_1 \geqq F_2)$.

Theorem 3. For any $X \leqq E$, $X^{*\prime}$ is the (strong) closure $\overline{conv}(X, 0)$ of the convex hull $conv(X, 0)$ of X and the zero vector 0 of E.

Proof. Surely we have $X^{*\prime} \geqq \overline{\text{conv}}(X, 0)$. If $x_0 \in X^{*\prime} - \overline{\text{conv}}(X, 0)$, there exists, by Ascoli-Mazur's theorem, an $f_0 \in \bar{E}$ such that $f_0(x_0) > 1$ and $f_0(x) \leqq 1$ at every $x \in \overline{\text{conv}}(X, 0)$. Then $f_0 \in X^*$ and hence x_0 must be $\bar{\in} X^{*\prime}$, contrary to the hypothesis. Thus we must have $X^{*\prime} = \overline{\text{conv}}(X, 0)$.

Theorem 4. For any $F \leqq \bar{E}$, $F^{\prime *}$ is the closure $\overline{\text{conv}}(F, 0)$ (in the sense of weak topology of $\bar{E}$ as functionals) of the convex hull $\overline{\text{conv}}(F, 0)$ of F and the zero vector 0 of $\bar{E}$.

Proof. Surely we have $F^{\prime *} \geqq \overline{\text{conv}}(F, 0)$. If $f_0 \in F^{\prime *} - \overline{\text{conv}}(F, 0)$, then there exists, by the definition of weak topology, $\varepsilon > 0$ and x_1, $x_2, \dots, x_n, \in E$ such that $\sup_{1 \leqq i \leqq n} |f(x_i) - f_0(x_i)| > \varepsilon$ at every $f \in \overline{\text{conv}}(F, 0)$. Thus the point $\xi_{f_0} = (f_0(x_1), f_0(x_2), \dots, f_0(x_n))$ in n-dimensional euclidian space E_n is of euclidian distance $> \varepsilon$ from the convex point set $\{\xi_f\}$ in E_n, where $\xi_f = (f(x_1), f(x_2), \dots, f(x_n))$, $f \in \overline{\text{conv}}(F, 0)$. Hence, by Ascoli-Mazur's theorem, there exists real numbers $a_1, a_2, \dots, a_n$ such that $\sum_{i=1}^{n} a_i f_0(x_i) > 1$ and $\sum_{i=1}^{n} a_i f(x_i) \leqq 1$ at every $f \in \overline{\text{conv}}(F, 0)$. Put $x_0 = \sum_{i=1}^{n} a_i x_i$, then $x_0 \in F'$ by $f(x_0) \leqq 1$. This contradicts to $1 < f_0(x_0)$, $f_0 \in F^{\prime *}$. Therefore we must have $F^{\prime *} = \overline{\text{conv}}(F, 0)$.

By Lemma 1, Theorem 1, 3 and 4 we obtain the following duality

Theorem 5. Regularly convex set $F \leqq \bar{E}$ containing 0 is characterised by the property that it is expressible as $F = X^*$, $X \leqq E$. (Strongly) closed convex set $X \leqq E$ containing 0 is characterised by the property that it is expressible as $X = F'$, $F \leqq \bar{E}$.

§ 3. *Boundedness of convex sets and regularly convex sets.* Regularly convex set F need not be (strongly) bounded. However we may prove

Theorem 6. In order that a regularly convex set F be (strongly) bounded, it is necessary and sufficient that 0 is the inner point of the (strongly) closed convex set $F' \leqq E$.

Proof. Let $\sup_{f \in F} \|f\| = \alpha < \infty$, and assume that 0 is not an inner point of F'. Then we have a sequence $\{x_i\} \leqq E$ with the properties

1) Cf. Garrett Birkhoff: Lattice Theory, New York (1940), 24.

$x_i \bar{\in} F'(i=1, 2, \dots)$, $\lim_{i\to\infty} \|x_i\|=0$. This contradicts to $|f(x_i)| \leqq \|f\|\cdot\|x_i\| \leqq \alpha\|x_i\|$, $(i=1, 2, \dots)$, $f\in F$. For we have $|f(x_i)|\leqq 1$, viz. $x_i \in F'$ if $\|x_i\|\leqq \frac{1}{\alpha}$. Next let the sphere $\|x\|\leqq \alpha$, $\alpha>0$, of E be $\leqq F'$, and assume that F is not (strongly) bounded. Then there exists $f_0 \in F$ such that $f_0(x_0)>1$ or $f_0(x_0)<-1$ at a point x_0 with $\|x_0\|=\alpha$. Since $\|-x_0\|=\|x_0\|$, x_0 and $-x_0$ both $\in F'$, and thus we must have a point $x_1 \in F'$ such that $f_0(x_1)>1$. This contradicts to the fact that $f_0\in F$.

Dually we may prove

Theorem 7. In order that a regularly convex set F contains 0 *as an inner point, it is necessary and sufficient that the* (*strongly*) *closed convex set F' be* (*strongly*) *bounded.*

Proof. Let $\sup_{x\in F'} \|x\|=\alpha$, $0<\alpha<\infty$, then $f\in\bar{E}$ with $\|f\|\leqq\frac{1}{\alpha}$ surely belongs to $F'^{*}=F$. Next let the sphere $\|f\|\leqq \alpha$, $0<\alpha<\infty$, be contained in F. By Hahn-Banach's theorem, there exists, for any $x_0\in E$, an $f_0 \in \bar{E}$ such that $\|f_0\|=1$, $f_0(x_0)=\|x_0\|$. Thus we must have $\|x\|\leqq\frac{1}{\alpha}$ for any $x\in F'$.

§ 4. *Existence of extreme points.* Let E be separable and let a (strongly) closed convex set $X\leqq E$ contain 0 as an inner point. Then, by S. Mazur's theorem cited in § 1, there exists boundary point x_0 of X at which we have one and only one tangential hyperplane. This means that there exists one and only one $f_0 \in \bar{E}$ with the properties $f_0(x_0)=1$, $\sup_{x\in X} f_0(x)\leqq 1$. Such f_0 is surely an extreme point of the (strongly) bounded regularly convex set X^*. Hence, combined with theorem 5 and 6, we obtain a new proof of Theorem 2 in the special case when E is separable.

In order to prove Theorem 2 for non-separable E we first prove the

Lemma 2. A (*strongly*) *bounded regularly convex set $F\leqq\bar{E}$ is bicompact in the weak topology of $\bar{E}$ as functionals.*

Proof. Let $\sup_{f\in F}\|f\|=\alpha$, $\alpha<\infty$. The sphere $G: \|g\|\leqq\alpha$ of $\bar{E}$ is bicompact in the weak topology of $\bar{E}$ as functionals[1]. Since, by Theorem 1, $F\leqq G$ is closed in the weak topology of $\bar{E}$ as functionals, F is bicompact with G.

Proof to Theorem 2. Let, as above, $\sup_{f\in F}\|f\|=\alpha$, $0<\alpha<\infty$, where F is regularly convex. Well-order the points of the unit-sphere $\|x\|\leqq 1$ of E as follows:

$$(1) \qquad x_0, x_1, x_2, \cdots, x_\alpha, \cdots \qquad (\alpha<\eta).$$

F is bicompact in the weak topology of $\bar{E}$ as functionals, by Lemma 2. Hence, for any $x\in E$, the continuous function $f(x)$ on $F(f\in F)$ attains its supremum and infimum on F. Let $\sup_{f\in F} f(x_0)=f_1(x_0)$, $f_1\in F$,

1) See S. Kakutani: loc. cit.

and let F_1 be the totality of such $f_1 \in F$. F_1 is convex and closed (in the weak topology of $\bar{E}$ as functionals), and is thus bicompact. If F_1 contains only one point f_1^0, f_1^0 is surely an extreme point of F and the Theorem 2 is proved. Assume that F_1 consists of more than one point, and let x_a be the first point in the well-ordered sequence (1) which satisfies $f(x_a) \neq$ constant on F_1. Put $a = a_1$ and let $\sup_{f_1 \in F_1} f_1(x_{a_1}) = f_2(x_{a_1})$, $f_2 \in F_1$, and denote by F_2 the totality of such $f_2 \in F_1$. In this way let possibly transfinite sequence of convex, bicompact sets $F \geqq F_1 \geqq F_2 \geqq \cdots \geqq F_\xi \geqq \cdots$ be defined for all $\xi < \xi_1 (< \eta)$. If ξ_1 is not a limit ordinal, let $a_{\xi_1 - 1}$ be the least a such that $f(x_a) \neq$ constant on $F_{\xi_1 - 1}$. Let $\sup_{f \in F_{\xi_1 - 1}} f(x_{a_{\xi_1 - 1}}) = f_{\xi_1}(x_{a_{\xi_1 - 1}})$, $f_{\xi_1} \in F_{\xi_1 - 1}$, and define F_{ξ_1} as the totality of such $f_{\xi_1} \in F_{\xi_1 - 1}$. If ξ_1 is a limit ordinal, put $F'_{\xi_1} =$ the intersection $\bigwedge_{\xi < \xi_1} F_\xi$. As F_ξ are all bicompact, F'_{ξ_1} is not void. We then define F_{ξ_1}, taking F'_{ξ_1} in place of $F_{\xi_1 - 1}$ in the above argument.

Thus, assuming the process not to give any extreme point of F, we define bicompact sets F_ξ for all $\xi < \eta$ such that: $F \geqq F_1 \geqq \cdots \geqq F_\xi \geqq \cdots$. Then $\bigwedge_{\xi < \eta} F_\xi$ is not void. $\bigwedge_{\xi < \eta} F_\xi$ consists of only one point. For, if $f, g \in \bigwedge_{\xi < \eta} F_\xi$, $f \neq g$, then $f(x) \neq g(x)$ for a certain $x = x_a$, $a < a_\xi$. This contradicts $f(x_a) = g(x_a)$ (by $f, g \in F_\xi$). Thus let $f_0 = \bigwedge_{\xi < \eta} F_\xi$, f_0 is an extreme point of F. For the proof, take any two $g, h \in F$, $g \neq f_0$, $h \neq f_0$. Then there exists the least ξ such that g, h both $\in F_\xi$. Thus $g(x_a)$, $h(x_a)$ both $< f_0(x_a)$ for all $a > a_\xi$. This proves that $f_0 \neq \frac{1}{2}(g + h)$.