4. *A Remark on Kawakami's Extension of Löwner's Lemma.*

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I. Recently M. Y. Kawakami, using Brelot's "hypofonction"[1], has proved the following theorem, which is an extension of Löwner's Lemma[2].

Theorem of Kawakami. *A domain D with boundary C in the unit circle $|w|<1$ in the w-plane is represented on the unit circle $|z|<1$ in the z-plane by a simple function $z=F(w)$ such that $F(0)=0$. If a closed set E' on C on $|w|=1$ is represented on a closed set E on $|z|=1$, then we have*

$$\text{measure of } E \leqq \text{measure of } E'.$$

And the equality holds only when D coincides with the unit circle $|w|<1$, if E' is of positive measure.

In this short note, we shall prove the first half of the above results under remarkably general conditions.

II. Let $f(z)$ be a regular, bounded function $|f(z)|<1$ defined for $|z|<1$. Let E be any set measurable or not on $|z|=1$ such that for almost every point $e^{i\theta}$ of E, $\lim_{r\to 1-0} f(re^{i\theta})=f(e^{i\theta})$ exists and satisfies the condition $|f(e^{i\theta})|=1$. Then we shall denote by E' the set of such boundary values $f(e^{i\theta})$. We shall further denote by m^*E, m_*E and mE the outer measure, the inner measure and the measure of E respectively, measured by the arc-length along the unit circle.

Now the theorem we are to prove is as follows:

Theorem. *If $w=f(z)$ is an analytic function such that $|f(z)|<1$ for $|z|<1$ and $f(0)=0$, then we have*

$$m_*E \leqq m^*E'.$$

Proof. Given an arbitrary positive number ε, we can choose an enumerable sequence of open arcs on the boundary $|w|=1$, such that the sum O of the sets of the sequence contains the set E' and

$$mO < m^*E' + \varepsilon. \tag{1}$$

Let $\underline{E}$ be the measurable subset of E such that $m\underline{E}=m_*E$[3]. We shall denote by $C(\varphi)$ and $C'(\psi)$ the characteristic functions, defined for $z=e^{i\varphi}$,

1) M. Brelot, Familles de Perron et Problème de Dirichlet. Acta Litt. ac. sc. un. Hungariae, (1939), 133–153.

2) Y. Kawakami, On an extension of Löwner's lemma. Japan. Jour. of Math. Vol. **17**, (1941), 569–572.

3) C. Carathéodory, Vorlesungen über reelle Funktionen, Zweite Auflage, (1927), p. 261, Theorem 3.

$0 \leqq \varphi \leqq 2\pi$ and $w = e^{i\psi}$, $0 \leqq \psi \leqq 2\pi$, of the sets $\underline{E}$ and O respectively. Now consider the following Poisson's expressions

$$(2) \qquad u(z) = u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\varphi-\theta)} \cdot C(\varphi)\, d\varphi\,,$$

$$(3) \qquad U(w) = U(Re^{i\Theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-R^2}{1+R^2-2R\cos(\psi-\Theta)} \cdot C'(\psi)\, d\psi\,,$$

of which both are harmonic and whose values are $\leqq 1$ and $\geqq 0$.

Denoting $U(f(z)) - u(z)$ by $V(z)$, which is harmonic and bounded for $|z| < 1$, from the well known theorem of Fatou[1], $V(z)$ is expressed by the following Poisson's formula,

$$V(z) = V(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\varphi-\theta)}\, V(\varphi)\, d\varphi\,,$$

where $V(\varphi) = \lim\limits_{r \to 1} V(re^{i\varphi})$ for almost all φ.

By the continuity of $C'(\psi)$ at the point $e^{i\psi}$ of the open set O, $U(w)$ is also continuous at that point up to the boundary, so that we have, from (3) for almost all points $e^{i\varphi}$ of the set $\underline{E}$,

$$(4) \qquad \lim_{r\to 1} U(f(re^{i\varphi})) = \lim_{w \to e^{i\psi}} U(w) = C'(\psi) = 1\,,$$

where $\lim\limits_{r\to 1} f(re^{i\varphi}) = e^{i\psi}$, and on the other hand, from (2)

$$(5) \qquad \lim_{r\to 1} u(re^{i\varphi}) = C(\varphi) = 1\,.$$

Similarly, for almost all points of the complementary set of $\underline{E}$, we have

$$(6) \qquad \lim_{r\to 1} U(f(re^{i\varphi})) \geqq 0\,, \qquad \lim_{r\to 1} u(re^{i\varphi}) = C(\varphi) = 0\,.$$

From (4), (5) and (6), we have $V(\varphi) \geqq 0$ for almost all φ $0 \leqq \varphi \leqq 2\pi$, and consequently $V(re^{i\varphi}) \geqq 0$ for $r < 1$, which shows

$$U(f(re^{i\varphi})) \geqq u(re^{i\varphi})\,.$$

Putting $r = 0$ in the relation above, it follows immediately that

$$mO \geqq m\underline{E}\,,$$

since $U(0)$ and $u(0)$ are the mean values of the boundary values of $U(w)$ and $u(z)$ respectively.

It follows, from (1),

$$m^*E' + \varepsilon \geqq m_*E\,.$$

Hence, ε being arbitrary, we have

$$m^*E' \geqq m_*E\,.$$

Q. E. D.

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1) P. Fatou, Séries trigonométriques et séries de Taylor. Acta Math. **30**, (1906), 335–400.