39. *On Krull's Conjecture Concerning Completely Integrally Closed Integrity Domains. I.*

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In the important papers, *Allgemeine Bewertungstheorie*, Crelles Journal 167 (1932) and *Beiträge zur Arithmetik kommutativer Integritätsbereiche*, Math. Zeitschr. 41 (1936), W. Krull gave a conjecture[1] that every completely integrally closed (= vollständig ganz-abgeschlossen)[2] integrity domain can always be expressed, in its quotient field, as an intersection of special valuation rings[3]. On ignoring addition A. H. Clifford has worked on the problem whether or not every Archimedean partially ordered abelian group can be embedded in a real component vector group, or what is the same, represented faithfully by (finite) real-valued functions[4]. In the following we want to show that the conjectures can not be the case in general. We shall first take up the simpler case of partially ordered abelian groups; The case of integrity domains will be treated in Part II.

Now, let A be a complete Boolean algebra and $\Omega=\Omega(A)$ be its representation space, that is, the totality of prime dual ideals of A with Stone-Wallman's topology; when $a\in A$ the so-called a-set, the set of prime dual ideals containing a, is an open and closed subset of Ω, and conversely every open and closed subset of Ω is an a-set; the system of all the a-sets forms a basis of closed sets in $\Omega:\Omega$ is thus a totally disconnected bicompact T_1-space. In Ω Borel sets coincide with open and closed sets (a-sets) *mod.* sets of first category. From this follows, as T. Ogasawara pointed out recently[5], that in Ω every Borel-measurable function finite except on a set of first category coincides except on a set of first category with a (real and $\pm\infty$-valued) continuous function finite except on a nowhere dense set, and the totality of the functions of the last class, namely (real and $\pm\infty$-valued) continuous functions on Ω finite except on nowhere dense sets, forms a vector-lattice $\mathfrak{L}_\Omega=\mathfrak{L}_{\Omega(A)}$. The order relation in $\mathfrak{L}_\Omega$ is point-wise as usual. As for addition it is as follows: the sum $g+h$ of two elements g, h in $\mathfrak{L}_\Omega$ is the continuous function on Ω finite except on a nowhere dense set coinciding with the

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1) § 4 and Part II, § 1, respectively, of the cited papers by W. Krull. Cf. also P. Lorenzen, Abstrakte Begründung der multiplikativen Idealtheorie, Math. Zeitschr. **45** (1939).

2) An integrity domain I is called completely integrally closed when an element x in its quotient field such that $x^n a\in I$ $(n=1,2,\dots)$ for a suitable element $a\ (\neq 0)$ in I lies necessarily in I.

3) An (exponential) valuation is called special when its value group consists of real numbers.

4) A. H. Clifford, Partially ordered abelian groups, Ann. Math. **41** (1940).

5) T. Ogasawara, On Boolean spaces (in Japanese), Zenkoku-Sizyo-Sugaku-Danwakai **230** (1941).

function-sum $g(\mathfrak{p})+h(\mathfrak{p})$ except on a set of first category. We notice here that for such a point $\mathfrak{p}$ of Ω that the sum $g(\mathfrak{p})+h(\mathfrak{p})$ is not indefinite the value $(g+h)(\mathfrak{p})$ of $g+h$ is equal to $g(\mathfrak{p})+h(\mathfrak{p})$. In particular this applies to those points where both $g(\mathfrak{p})$ and $h(\mathfrak{p})$ are finite.

Evidently $\mathfrak{L}_\Omega$ is Archimedean (and even complete).

Lemma 1. Assume that in our complete Boolean algebra A there exists a countable set of non-atomic non-zero elements $v_1, v_2, \dots, v_i, \dots$ such that for any $a>0$ in A we have $a \geqq v_i$ for a suitable i. Then for every point $\mathfrak{p}$ in Ω there exists always an element in $\mathfrak{L}_\Omega$, that is, a continuous function on Ω finite except on a nowhere dense set, which actually takes the value $+\infty$ at $\mathfrak{p}$. (The assumption of this lemma is fulfilled, for instance, by a complete Boolean algebra of regular open sets of the interval $(0, 1)$[1].)

Proof. Consider an arbitrary point $\mathfrak{p}$ in Ω, that is, a prime dual ideal in A. If $\mathfrak{p}=\{w\}$ then evidently $\inf w=0$. There exists hence w_1 in $\mathfrak{p}$ such that $w_1 \not\geqq v_1$. Further, there is w_2 in $\mathfrak{p}$ such that $w_1 \geqq w_2$ and $w_2 \not\geqq v_2$. Proceeding in this way we obtain a monotonic sequence

$$w_1 \geqq w_2 \geqq \cdots \geqq w_i \geqq \cdots$$

of elements in $\mathfrak{p}$ such that $w_i \not\geqq v_i$ for $i=1, 2, \dots$. Then evidently $\inf w_i=0$.

The corresponding open closed sets

$$w_1\text{-set} \geqq w_2\text{-set} \geqq \cdots \geqq w_i\text{-set} \geqq \cdots (\ni \mathfrak{p})$$

in Ω possess as the intersection a closed set $\frown(w_i\text{-set})$ which is nowhere dense and $\ni \mathfrak{p}$. Define $f(\mathfrak{q})$ as follows:

$$\begin{aligned} f(\mathfrak{q})&=0 && \text{when } \mathfrak{q} \notin w_1\text{-set}, \\ f(\mathfrak{q})&=i && \text{when } \mathfrak{q} \in (w_i\text{-set}) - (w_{i+1}\text{-set}), \\ f(\mathfrak{q})&=+\infty && \text{when } \mathfrak{q} \in w_i\text{-set for all } i=1, 2, \dots. \end{aligned}$$

Then f is continuous, since w_i-sets are open and closed, and is finite except on the nowhere dense set $\frown(w_i\text{-set})$. This proves the lemma.

Lemma 2. Assume that for every point $\mathfrak{p}$ in $\Omega=\Omega(A)$ there exists an element f in $\mathfrak{L}=\mathfrak{L}_\Omega$ which actually assumes the value $+\infty$ at $\mathfrak{p}$. Then the vector-lattice $\mathfrak{L}=\mathfrak{L}_\Omega$ can never be represented faithfully by (finite) real-valued functions.

Proof. Denote the set of elements g in $\mathfrak{L}$ finite at a point $\mathfrak{p}$ by $\mathfrak{N}_\mathfrak{p}$. $\mathfrak{N}_\mathfrak{p}$ is a normal subspace[2] of $\mathfrak{L}$, and does not coincide with $\mathfrak{L}$ because of our assumption. The intersection $\mathfrak{N}=\frown\mathfrak{N}_\mathfrak{p}$ of all the $\mathfrak{N}_\mathfrak{p}$, $\mathfrak{p}$ running over Ω, is nothing but the set of all elements in $\mathfrak{L}$ finite everywhere.

Every homomorphic mapping of $\mathfrak{L}$ upon the ordered group R of real number is obtained from a suitable maximal normal subspace $\mathfrak{M}$ of $\mathfrak{L}$; $\mathfrak{L}/\mathfrak{M} \cong R$. Thus, consider a maximal normal subspace $\mathfrak{M}$. There

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1) See for instance G. Birkhoff, Lattice theory, New York 1940, § 124.

2) In the sense of G. Birkhoff; m-subgroup in the sense of T. Nakayama, Note on lattice-ordered groups, Proc. **17** (1941).

are two possibilities, which can be, at least, thought of: 1) There exists a $\mathfrak{p} \in \Omega$ such as $\mathfrak{N}_{\mathfrak{p}} \geqq \mathfrak{M}$: 2) There is no such $\mathfrak{p}$. In the case 1) we have, since $\mathfrak{M}$ is maximal, necessarily $\mathfrak{N}_{\mathfrak{p}} = \mathfrak{M}$ whence $\mathfrak{N} = \frown \mathfrak{N}_{\mathfrak{p}} \leqq \mathfrak{M}$. In the case 2) there exists for each $\mathfrak{p} \in \Omega$ an element $f_{\mathfrak{p}}$ in $\mathfrak{M}$ such that $f_{\mathfrak{p}}(\mathfrak{p}) = +\infty$. Let $U_{\mathfrak{p}}$ be a neighborhood of $\mathfrak{p}$ such that $\mathfrak{q} \in U_{\mathfrak{p}}$ implies $f_{\mathfrak{p}}(\mathfrak{q}) > 1$, say. Since Ω is bicompact there is a finite system $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_n$ of points in Ω so that Ω is covered by $U_{\mathfrak{p}_1}, U_{\mathfrak{p}_2}, \cdots, U_{\mathfrak{p}_n}$. Put then

$$F = f_{\mathfrak{p}_1} \cup f_{\mathfrak{p}_2} \cup \cdots \cup f_{\mathfrak{p}_n}.$$

F is contained in $\mathfrak{M}$. If g is an element in $\mathfrak{N}$, there is a natural number m such as $|g| \leqq mF$; we have only to choose m greater than the maximum value of g over Ω. Therefore, again $\mathfrak{N} \leqq \mathfrak{M}$.

Since this is the case for every maximal normal subspace $\mathfrak{M}$, the intersection of all the maximal normal subspace of $\mathfrak{L}$ contains $\mathfrak{N}$. This means that the elements in $\mathfrak{N}$ are mapped onto zero in any homomorphic mapping of $\mathfrak{L}$ upon R. But evidently $\mathfrak{N} \neq 0$, and thus $\mathfrak{L}$ can never be represented isomorphically by (finite) real-valued functions.

Combining these lemmas we get

Theorem. Let A be a complete Boolean algebra satisfying the condition of Lemma 1; for instance, A may be the Boolean algebra of regular open sets on (0, 1). *Then the vector-lattice* $\mathfrak{L} = \mathfrak{L}_{\Omega}$ $\big(\Omega = \Omega(A)\big)$ *is Archimedean but can never be represented faithfully by* (*finite*) *real-valued functions.*

Remark 1. As a matter of fact, the case 1) can not occur. For $\mathfrak{N}_{\mathfrak{p}}$ is never a maximal subspace. Indeed, there is a linearly ordered system of continuum-many distinct normal subspaces between $\mathfrak{L}$ and $\mathfrak{N}_{\mathfrak{p}}$, corresponding to distinct orders of infinity at $\mathfrak{p}$, roughly speaking. From this observation we can, on modifying the above proof slightly, prove that in order to represent our $\mathfrak{L} = \mathfrak{L}_{\Omega}$ faithsully by functions taking values from a certain linearly ordered abelian group R_1, the system of distinct ranks[1] in R_1 has to have at least the power of continuum.

Remark 2. As Mr. H. Nakano has kindly pointed out, our proof applies for instance to the usual L_p (when represented in Nakano's fashion[2]).

Remark 3. Instead of considering the whole vector-lattice $\mathfrak{L}_{\Omega}$, we could restrict ourselves to continuous functions on Ω taking rational integers and $\pm\infty$ as values and finite except on nowhere dense sets. For, the function constructed in Lemma 1 is indeed such. The partially ordered abelian group thus obtained is Archimedean and can not be represented isomorphically by finite real-valued functions. This remark will be useful in Part II.

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1) In the sense of H. Hahn, Über die nichtarchimedischen Grössensysteme, Sitz.-ber. d. Math.-Nat. Klasse d. Wiener Akad. **116** II a (1907).

2) H. Nakano, Eine Spektraltheorie, Proc. Phys.-math. Soc. Japan **23** (1941). Cf. also the writer's note, On the representations of vector-lattices (in Japanese), Zenkoku-Sizyo-Sugaku-Danwakai **233** (1942).