46. On an Extension of Löwner's Theorem.

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We will prove the following extension of Löwner's theorem.

Theorem. *Let* $w=f(z)$ *be regular and* $|f(z)|<1$ *in* $|z|<1$, $f(0)=0$ *and* $\lim_{r\to 1} f(re^{i\theta})=e^{i\psi}$ *exists, when* θ *belongs to a set* E *and the* ψ*-set on* $|w|=1$ *be denoted by* E^*. *Then* E *and* E^* *are measurable and*

$$mE \leqq mE^* . \tag{1}$$

If $0<mE<2\pi$, *then* $mE<mE^*$.

Mr. Y. Kawakami[1] proved (1) under the condition that $f(z)$ is schlicht in $|z|<1$ and Messrs. S. Kametani and T. Ugaheri[2] proved that $m_iE\leqq m_eE^*$, where m_iE and m_eE denote the inner and outer measure of E.

Proof. Since $f(re^{i\theta})$ $(0<r<1)$ is continuous in $0\leqq\theta\leqq 2\pi$, by H. Hahn's theorem[3], the set e, where $\lim_{r\to 1} f(re^{i\theta})=\rho(\theta)e^{i\psi(\theta)}$ exists, is $F_{\sigma\delta}$, so that $\rho(\theta)$ and $\psi(\theta)$ are Borel functions defined on a Borel set e and hence the sub-set E of e, where $\rho(\theta)=1$, is a Borel set. Consider on the (θ, ψ)-plane a set M, whose points are $(\theta, \psi(\theta))$, where $\theta\in E$. We will prove that M is a Borel set on the (θ, ψ)-plane.

Let $0=a_0<a_1<\cdots<a_{n-1}<a_n=2\pi$, $a_k-a_{k-1}=\dfrac{1}{n}$ $(1\leqq k\leqq n)$ and

$$E_k=E\left(a_{k-1}\leqq\psi(\theta)\leqq a_k\right),$$

$$\underline{M}_k=\text{the set of points } (\theta, \psi), \text{ where } \theta\in E_k,\quad 0\leqq\psi<a_{k-1},$$

$$\underline{M}(n)=\sum_{k=1}^{n}\underline{M}_k ,$$

and

$$\overline{M}_k=\text{the set of points } (\theta, \psi), \text{ where } \theta\in E_k,\quad 0\leqq\psi\leqq a_k,$$

$$\overline{M}(n)=\sum_{k=1}^{n}\overline{M}_k .$$

Then for $n\to\infty$, $\underline{M}(n)\to\underline{M}$, $\overline{M}(n)\to\overline{M}$, so that $M=\overline{M}-\underline{M}$. Since $\overline{M}(n)$, $\underline{M}(n)$ are Borel sets, $\overline{M}$ and $\underline{M}$ and hence M is a Borel set. E^*, being the projection of M on the ψ-axis, is an analytic set, so that is measurable.

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1) Y. Kawakami; On an extension of Löwner's lemma. Japan. Jour. of Math. **17** (1941).

2) S. Kametani and T. Ugaheri: A remark on Kawakami's extension of Löwner's lemma. Proc. **18** (1942), 14.

3) Hausdorf. Mengenlehre, p. 271.

From this we can proceed similarly as Kametani-Ugaheri's proof. Let

$$u(z)=u(re^{i\theta})=\frac{1}{2\pi}\int_E \frac{1-r^2}{1-2r\cos(\varphi-\theta)+r^2}\,d\varphi\,,$$

$$U(w)=U(\rho e^{i\psi})=\frac{1}{2\pi}\int_{E*} \frac{1-\rho^2}{1-2\rho\cos(\varphi-\psi)+\rho^2}\,d\varphi\,,$$

$v(z)=U\big(f(z)\big)-u(z)$. Let O be an open set which contains E^*, $U_1(w)$ be the Poisson integral formed with O instead of E^* and $v_1(z)=U_1\big(f(z)\big)-u(z)$, then $\lim\limits_{r\to1} v_1(re^{i\theta})=0$ almost everywhere on E, $\geqq 0$ almost enerywhere on E' (the complementary set of E), so that $v_1(z)\geqq 0$ in $|z|<1$. Making $mO\to mE^*$, we have $v(z)\geqq 0$ in $|z|<1$. Hence $v(0)=mE^*-mE\geqq 0$, or $mE^*\geqq mE$. If $0<mE<2\pi$, then $0<mE\leqq mE^*$, so that

$$U(w)>0 \text{ in } |w|<1\,, \tag{2}$$

if in this case, $mE=mE^*$, then $v(0)=0$, so that $v(z)\equiv 0$, or

$$u(z)\equiv U\big(f(z)\big)\,. \tag{3}$$

Since $mE'>0$, by Fatou's theorem, there exists θ_0 in E', such that $\lim\limits_{r\to1} u(re^{i\theta_0})=0$, $\lim\limits_{r\to1} f(re^{i\theta_0})=w_0$ ($|w_0|<1$). Hence we have from (3), $U(w_0)=0$, which contradicts (2). Hence if $0<mE<2\pi$, then $mE<mE^*$.