PAPERS COMMUNICATED

45. On the Behaviour of a Meromorphic Function in the Neighbourhood of a Closed Set of Capacity Zero.

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1. *Nevanlinna's fundamental theorems.*

Let E be a bounded closed set of capacity 0 on the z-plane, which is contained in a bounded domain D and $w=w(z)=f(z)$ be meromorphic in $D-E$ and have every point of E as an essential singularity. Since E is of capacity 0, by Evans' theorem[1], we can distribute a positive mass $d\mu(a)$ on E, such that

$$u(z)=\int_E \log\frac{1}{|z-a|}d\mu(a)\,,\qquad \int_E d\mu(a)=1\,, \tag{1}$$

is harmonic in $D-E$ and $u(z)=\infty$ at every point of E. Let $\theta(z)$ be the conjugate harmonic function of $u(z)$ and put

$$t=e^{u(z)+i\theta(z)}=r(z)e^{i\theta(z)}\,. \tag{2}$$

This $r(z)$ plays the similar rôle as $|z|$ in the theory of meromorphic functions for $|z|<\infty$. Let C_r be the niveau curve: $r(z)=\text{const.}=r$, then C_r consists of finite number of closed curves surrounding E. We remark that $\int_{C_r} d\theta(z)=\int_{C_r}\frac{\partial u}{\partial n}ds=2\pi\int_E d\mu=2\pi$, where n is the inner normal of C_r. We assume that D is bounded by an analytic Jordan curve C and the domain bounded by C and C_r be denoted by Δ_r. Let K be the Riemann sphere of diameter 1, which touches the w-plane at $w=0$ and put $[a,b]=\frac{|a-b|}{\sqrt{(1+|a|^2)(1+|b|^2)}}$, $n(r,a)=$the number of zero points of $f(x)-a$ in Δ_r,

$$N(r,a)=\int_{r_0}^{r}\frac{n(r,a)}{r}dr\,,$$

$$m(r,a)=\frac{1}{2\pi}\int_{C_r}\log\frac{1}{[w(z),a]}d\theta(z)\,,$$

$$T(r,a)=m(r,a)+N(r,a)\,,$$

$A(r)=$the area on K, which is covered by $w=f(z)$, when z varies in Δ_r and $S(r)=\frac{A(r)}{\pi}$.

1) Evans: Potentials and positively infinite singularities of harmonic functions. Monathefte f. Math. u. Phys. **43** (1936).

Then the following theorem corresponds to Nevanlinna's first fundamental theorem.

Theorem I. $T(r, a)=T(r)+O(\log r)$,

where $$T(r)=\int_{r_0}^{r}\frac{S(r)}{r}dr\,.$$

Proof. Considering $f(z)$ as an analytic function of t,

$$\frac{\partial m(r,a)}{\partial r}-\frac{\partial m(r,b)}{\partial r}=\frac{1}{2\pi}\int_{C_r}\frac{\partial}{\partial r}\log\left|\frac{w-b}{w-a}\right|d\theta$$

$$=\frac{1}{2\pi r}\int_{C_r}d\arg\frac{w-b}{w-a}=\frac{1}{2\pi r}\int_{C_r+C}d\arg\frac{w-b}{w-a}-\frac{1}{2\pi r}\int_{C}d\arg\frac{w-b}{w-a}$$

$$=\frac{n(r,b)-n(r,a)}{r}+O\left(\frac{1}{r}\right),\quad \text{or}\quad T(r,a)=T(r,b)+O(\log r)\,.$$

Let $d\omega(b)$ be the surface element on K, then $T(r,a)=\frac{1}{\pi}\int_K T(r,b)d\omega(b)+O(\log r)=\int_{r_0}^{r}\frac{S(r)}{r}dr+O(\log r)$, q. e. d.

We will call $T(r)$ the characteristic function and $\overline{\lim\limits_{r\to\infty}}\frac{\log T(r)}{\log r}=\rho$ the order of $f(z)$ about E.

Theorem II. $\lim\limits_{r\to\infty}\frac{T(r)}{\log r}=\infty$.

Proof. If $|f(z)-a|\geqq\rho>0$ in $D-E$, then $\frac{1}{f(z)-a}$ is bounded in $D-E$, so that $\frac{1}{f(z)-a}$ is regular[2] and hence $f(z)$ is meromorphic on E, contradictory to the hypothesis. Hence in $D-E$, $f(z)$ takes the values which are dense on K. Let z_0 be a point on E and $D_1\supset D_2\supset\cdots\supset D_n\to z_0$ be a sequence of domains tending to z_0 and e_n be the values omitted by $f(z)$ in D_n, then e_n is non-dense, so that $e=\sum_{n=1}^{\infty}e_n$ is of first category. Hence there exists a point a which does not belong to e. This a is taken by $f(z)$ infinitely many times about z_0, so that $\lim\limits_{r\to\infty}\frac{T(r,a)}{\log r}=\infty$ and by Theorem I, $\lim\limits_{r\to\infty}\frac{T(r)}{\log r}=\infty$, q. e. d.

Let $\Lambda(r)$ be the number of closed boundary curves of Δ_r, then[3] $(q-2)S(r)\leqq\sum_{k=1}^{q}n(r,a_k)+\Lambda(r)+O\big(L(r)\big)$, where $L(r)$ is the length of the curve on K, which corresponds to C_r and $\lim\limits_{r\to\infty}\frac{L(r)}{S(r)}=0$. Similarly as for a meromorphic function for $|z|<\infty$, we can prove[3]

Theorem III. If $\overline{\lim\limits_{r\to\infty}}\frac{\Lambda(r)}{S(r)}=p<\infty$, *then* $f(z)$ *takes every value,*

2) R. Nevanlinna: Eindeutige analytische Funktionen. p. 132. Satz 2.

3) M. Tsuji: On the behaviour of an inverse function of a meromorphic function at its transcendental singular point. Proc. **17** (1941), 414.

except at most $p+2$ values, infinitely many times in $D-E$ and one of $q>2p+4$ disjoint simply connected domains on K is covered schlicht by the Riemann surface of the inverse function of $f(z)$.

Remark I. If $f(z)$ is regular in $D-E$ and $M(r)=\underset{C_r}{\text{Max.}}|f(z)|$, then $M(r)$ is an increasing function of r and $\log M(r)$ is a convex function of $\log r$.

II. Let e be a closed set of positive capacity, then we can distribute a positive mass $d\nu(a)$ on e, such that $\int_e \log\frac{1}{[w,a]}d\nu(a)$, $\int_e d\nu(a)=1$, is bounded on K. Hence by Theorem I,

$$T(r)=\int_e N(r,a)d\nu(a)+O(\log r). \tag{3}$$

This corresponds to Nevanlinna's second fundamental theorem.

III. *Theorem IV* (Noshiro)[4]. *Let $f(z)$ be regular and bounded in a bounded domain D and on the boundary of D, $\overline{\lim}|f(z)|\leqq 1$, except a closed set E of capacity 0, then $|f(z)|\leqq 1$ in D.*

Proof. Let $u(z)$ be the same as in (1) and put $v(z)=\log|f(z)|-\varepsilon u(z)$ $(\varepsilon>0)$. Since $u(z)=\infty$ on E and $f(z)$ is bounded, $\overline{\lim}\, v(z)\leqq 0$ on the boundary of D. Since $v(z)$ is sub-harmonic, $v(z)\leqq 0$ in D. Making $\varepsilon\to 0$, we have $|f(z)|\leqq 1$ in D.

2. *Applications.*

Theorem V. Let D be a domain bounded by a Jordan curve C and E be a closed set of capacity 0 contained in D and $f(z)$ be meromorphic in $D-E$ and have every point of E as an essential singularity. Then (i) *$f(z)$ takes every value a, except a-values of capacity 0, infinitely many times in $D-E$. More precisely $\overline{\lim\limits_{r\to\infty}}\frac{N(r,a)}{T(r)}=1$, except a-values of capacity 0.* (ii) *If further $f(z)$ be of finite order ρ and $z_n=z_n(a)$ be the zero points of $f(z)$-a and $r_n(a)=r(z_n)$, then $\sum\limits_{n=1}^{\infty}\frac{1}{[r_n(a)]^{\rho+\varepsilon}}$ $(\varepsilon>0)$ is convergent for all a, while $\sum\limits_{n=1}^{\infty}\frac{1}{[r_n(a)]^{\rho-\varepsilon}}$ is divergent, except a-values of capacity 0.*

The first part of (i) is due to Mr. S. Kametani[5].

Proof. (i) Let e be the set of values taken by $f(z)$ finite times in $D-E$ and suppose that e is of positive capacity and e_n be the subset of e, every value of which is taken by $f(z)$ at most n-times, then $e=\sum\limits_{n=1}^{\infty}e_n$. Hence one of e_n is of positive capacity, which as well known, contains a closed set of positive capacity. Hence we assume that e_n is a closed set of positive capacity. Then by (3), $T(r)=\int_{e_n}N(r,a)d\nu(a)+O(\log r)=O(\log r)$, which contradicts Theorem II. Hence e is of capacity

4) K. Noshiro: On the theory of the cluster sets of analytic functions. Jour. Faculty Science. Hokkaido Imp. Univ. **6** (1937–38).

5) S. Kametani: The exceptional values of functions with the set of capacity zero of essential singularities. Proc. **17** (1941), 429.

0. The second part can be proved similarly as Frostman[6]. (ii). Since the first part is evident, we will prove the second part. Suppose that $\sum_{n=1}^{\infty}\frac{1}{[r_n(a)]^{\rho'}}$ $(0<\rho'<\rho)$ is convergent, so that $\int_{r_0}^{\infty}\frac{N(r,a)}{r^{\rho'+1}}dr<\infty$ for a-values of positive capacity. Then as before there exists a closed set e_n of positive capacity, such that $\int_{r_0}^{\infty}\frac{N(r,a)}{r^{\rho'+1}}dr\leqq n$ for $a\in e_n$. Then by (3), $\int_{r_0}^{\infty}\frac{T(r)}{r^{\rho'+1}}dr=\int_{e_n}d\nu(a)\int_{r_0}^{\infty}\frac{N(r,a)}{r^{\rho'+1}}dr+O(1)=O(1)$, or $T(r)=O(r^{\rho'})$, which contradicts the hypothesis. Hence e is of capacity 0.

3. *An extension of Gross' theorem.*

We will prove the following extension of Gross' theorem.

Theorem VI. Under the same condition as Theorem V, let w_0 be a regular point of the inverse function $\varphi(w)$ of $f(z)$. Then we can continuate $\varphi(w)$ analytically along half-lines; $w-w_0=\rho e^{i\psi}$, till we meet the image Γ of C or indefinitely, except ψ-values of measure 0. *If $\varphi(w)$ be regular on a segment; $w-w_0=\rho$ $(0\leqq\rho_1<\rho<\rho_0)$, then starting from $w-w_0=\rho$, we can continuate $\varphi(w)$ analytically along circles; $w-w_0=\rho e^{i\psi}$ $(-\infty<\psi<\infty)$ indefinitely, except ρ-values of measure* 0.

Proof. For a given ψ, we continuate $\varphi(w)$ analytically along a half-line; $w-w_0=\rho e^{i\psi}$, till we meet Γ or a transcendental singularity of $\varphi(w)$, thus we get the principal star region H, with w_0 as its center. The edges of H are transcendental singularities of $\varphi(w)$. Let the part of H, which is contained in $|w-w_0|\leqq R$ be denoted by H_R, which corresponds to D_R on the z-plane. Let $t=e^{u+i\theta}=r(z)e^{i\theta(z)}$ be the same as in (2) and the part of the niveau curve C_r; $r(z)=r$, which lies in D_R be denoted by $C_r(R)$, which corresponds to σ_r in H_R, whose sum of lengths be denoted by $s(r)$, then putting $f(z)=F(t)$, we have $\left(s(r)\right)^2=\left(\int_{C_r(R)}|F'(t)|\,rd\theta\right)^2\leqq 2\pi r\int_{C_r(R)}|F'(t)|^2 rd\theta=2\pi r\frac{dA(r)}{dr}$, where $A(r)$ is the area of the part of H_R, which contains w_0 and is bounded by σ_r. Hence $\int_{r_0}^{r}\frac{\left(s(r)\right)^2}{r}dr\leqq 2\pi A(r)\leqq 2\pi^2R^2$. Since $r\to\infty$, we infer that there exists a sequence $r_n\to\infty$, such that $s(r_n)\to 0$. Hence the set of the ψ-values which correspond to the edges of H_R is of measure 0. Taking $R_1<R_2<\to\infty$ for R, we see that the set of ψ-values which correspond to the edges of H is of measure 0. The second part can be proved similarly. q. e. d.

From Theorem VI, we see that every boundary point of the Riemann surface F of $\varphi(w)$ is an accessible point. Hence if $f(z)\neq a$ in $D-E$, then a is a boundary point of F, so that there exists a curve Γ on F ending at a, which corresponds to a curve L on the z-plane ending at a point of E. Hence we have

Theorem VII (Cartwright-Noshiro)[7]. *Under the same condition as*

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6) Frostman: Potentiel d'equilibre et capacité des ensembles. Lund 1935.

7) Cartwright: On the asymptotic values of functions with a non-enumerable set of essential singularities. Jour. London Math. Soc. **11** (1936).

Noshiro, l. c. 4).

Theorem V, if $f(z) \neq a$ *in* $D-E$, *then there exists a curve* L *in* $D-E$ *ending at a point of* E, *such that* $f(z) \to a$ *along* L.

4. *Functions of class* (U).

The functions of class (U) are functions which satisfy the conditions: (i) $f(z)$ is regular and $|f(z)|<1$ in $|z|<1$. (ii) $|f(z)|=1$ almost everywhere on $|z|=1$.

Theorem VIII. Let $w=f(z)$ *belong to class* (U) *and* $\varphi(w)$ *be its inverse function defined in* $|w|<1$ *and* w_0 *be a regular point of* $\varphi(w)$. *Then we can continuate* $\varphi(w)$ *analytically along half-lines;* $w-w_0=\rho e^{i\psi}$, *till we meet* $|w|=1$, *except* ψ*-values of measure* 0. *Let* $\varphi(w)$ *be regular on a segment;* $w-w_0=\rho$ $(0 \leqq \rho_1 < \rho < \rho_0)$ *and* $|w-w_0|<\rho_0$ *be contained in* $|w|<1$. *Then we can continuate* $\varphi(w)$ *analytically along circles;* $w-w_0=\rho e^{i\psi}$ $(-\infty < \psi < \infty)$ *indefinitely, except* ρ*-values of measure* 0.

Proof. As in the proof of Theorem VI, let H_R be the part of the principal star region H with center at w_0, which is contained in a circle; $\left|\frac{w-w_0}{1-\bar{w}_0 w}\right| \leqq R < 1$. Let H_R correspond to D_R in $|z<1$. Then the set of transcendental singularities of $\varphi(w)$ which lie on the boundary of H_R corresponds to a closed set E of measure 0 on $|z|=1$. Let the set of open arcs $\{s_n\}$ be the complementary set of E on $|z|=1$. Then there exists[8] a positive function $F(\theta)$ on $|z|=1$, such that $F(\theta)$ is continuous on s_n and $F(\theta)=\infty$ on E and $\int_0^{2\pi} F(\theta)d\theta < \infty$. Let $u(z)$ be the Poisson integral with the boundary function $F(\theta)$, then $u(z)$ tends to infinity, when z tends to any point of E. Let $\theta(z)$ be the conjugate harmonic function of $u(z)$ and put $t=e^{u+i\theta}=r(z)e^{i\theta(z)}$. By means of this t, we can prove similarly as Theorem VI.

Similarly as Theorem VII, we have

Theorem IX (Seidel)[9]. *Let* $f(z)$ *belong to class* (U) *and* $f(z) \neq a$ $(|a|<1)$ *in* $|z|<1$, *then there exists a curve* L *ending on* $|z|=1$, *such that* $f(z) \to a$ *along* L.

5. *Inverse function of* $f(z)$ *in* § 1.

Let $w=w(z)=f(z)$ satisfy the conditions in § 1 and $z=\varphi(w)$ be its inverse function. A δ-neighbourhood U of w_0 on the Riemann surface F of $\varphi(w)$ is the connected part of F, which lies in $[w, w_0]<\delta$ and has w_0 as an inner point or a boundary point. Let U correspond to Δ on the z-plane, then $[f(z), w_0]<\delta$ in Δ and $[f(z), w_0]=\delta$ on the boundary of Δ, except the points on E. We assume that w_0 is an accessible boundary point of F, such that there exists a curve Γ on F ending at w_0, which corresponds to a curve L in Δ ending at z_0 on E. Let $t=e^{u+i\theta}=r(z)e^{i\theta(z)}$ be the same as in (2) and the part of Δ, such that $r(z) \leqq r$, $r(z)=r$ be noted by Δ_r, θ_r respectively. We put $A(r;\Delta)=$ the area on K, which is covered by $w=f(z)$, when z varies in Δ_r, $S(r;\Delta)=\frac{A(r;\Delta)}{\pi\delta^2}$, where $\pi\delta^2$ is the area of $[w, w_0] \leqq \delta$ on K, $n(r, a; \Delta)=$

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8) Fatou: Séries trigonométriques et séries de Taylor. Acta Math. **30** (1906).

9) Seidel: On the distribution of values of bounded analytic functions. Trans. Amer. Soc. **36** (1934).

the number of zero points of $f(z)-a$ in Δ_r, where $[a, w_0]<\delta$,

$$N(r, a; \Delta)=\int_{r_0}^{r}\frac{n(r, a; \Delta)}{r}dr,$$

$$m(r, a; \Delta)=\frac{1}{2\pi}\int_{\theta_r}\log\frac{1}{[w(z), a]}d\theta(z),$$

$$T(r, a; \Delta)=m(r, a; \Delta)+N(r, a; \Delta),$$

$L(r)=$ the length of the curve on K, which corresponds to θ_r.

Then exactly as I have proved[10] for a meromorphic function for $|z|<\infty$, we have

Theorem X. $T(r, a; \Delta)=T(r; \Delta)+O\left(\int_{r_0}^{r}\frac{L(r)}{r}dr\right)+O(\log r)$,

where $$T(r; \Delta)=\int_{r_0}^{r}\frac{S(r; \Delta)}{r}dr.$$

$$L(r)=O\left(\sqrt{T(2r; \Delta)\log r}\right) \quad \textit{for all } r, \tag{4}$$

$$L(r)=O\left(\sqrt{T(r; \Delta)}\log T(r; \Delta)\right), \tag{5}$$

except certain intervals I_n, *such that* $\sum_n\int_{I_n}d\log r<\infty$.

We will call $T(r; \Delta)$ the characteristic function and $\overline{\lim_{r\to\infty}}\frac{\log T(r; \Delta)}{\log r}=\rho$ the order of $f(z)$ in Δ.

Theorem XI. $\lim_{r\to\infty}\frac{T(r; \Delta)}{\log r}=\infty$.

Proof. We have two cases. (i) The branch of θ_r which meets L, always meets the boundary Λ of Δ. Let θ_r meets L at z' and Λ at z''. Since $u(z)=\infty$ on E, z'' is not a point of E, so that $f(z'')$ lies on $[w, w_0]=\delta$ and since $f(z')$ tends to w_0, we have $L(r)\geqq d>0$ for $r\geqq r_1$. Since $\int_{r_0}^{r}\frac{[L(r)]^2}{r}dr\leqq 2\pi A(r; \Delta)$[11], we have $\lim_{r\to\infty}A(r; \Delta)=\infty$. (ii) There exists a sequence of points $z_n\to z_0$ on L, such that the branch of θ_r, which meets L at z_n does not meet Λ, so that infinitely many disjoint boundary elements of Δ cluster at z_0. Hence the Riemann surface F of $\varphi(w)$ contains infinitely many sheets, $F_1, F_2, \dots$. By Theorem VI, the set e_n of the boundary points of F_n is non-dense, so that $e=\sum_{n=1}^{\infty}e_n$ is of first category, hence there exists a point a in $[w, w_0]<\delta$, which does not belong to e. This a is a regular point of $\varphi(w)$ on each F_n. Consider the principal star region H_n on F_n, with center at a. F can be considered as obtained by connecting H_n along the boundaries of H_n. By Theorem VI, the measure of H_n is $\pi\delta^2$. Since there are infinitely many sheets, $\lim_{r\to\infty}A(r; \Delta)=\infty$. From (i), (ii), we have

10) M. Tsuji: On the behaviour of an inverse function of a meromorphic function at its transcendental singular point, III. Proc. **18** (1942), 132.

11) M. Tsuji. l. c. 10).

$$\lim_{r\to\infty}\frac{T(r;\Delta)}{\log r}=\infty.$$

Theorem XII. Under the same condition as Theorem V, let w_0 be an accessible transcendental singularity of the inverse function $\varphi(w)$ of $f(z)$ and a δ-neighbourhood U of w_0 correspond to Δ on the z-plane. Then (i) *$f(z)$ takes every value a in $[w, w_0]<\delta$ infinitely many times in Δ, except a-values of capacity* 0. (ii) *If further $f(z)$ be of finite order ρ in Δ, then $\sum_{n=1}^{\infty}\frac{1}{[r_n(a)]^{\rho+\varepsilon}}$ $(\varepsilon>0)$ is convergent for all a in $[w, w_0]<\delta$, while $\sum_{n=1}^{\infty}\frac{1}{[r_n(a)]^{\rho-\varepsilon}}$ is divergent, except a-values of capacity* 0, *where $r_n(a)=r(z_n)$, z_n being the zero points of in Δ.*

Proof. Suppose that $f(z)$ takes in Δ finite times a-values which belong to a set e of positive capacity. Then as before $f(z)$ takes at most n-times a-values which belong to a closed set e_n of positive capacity. Then by Theorem X, (3) and (5), we have $T(r;\Delta)=\int_{e_n}N(r,a;\Delta)d\nu(a)$ $+O\Big(\int_{r_0}^{r}\frac{L(r)}{r}dr\Big)+O(\log r)=O\big(\sqrt{T(r;\Delta)}\log T(r;\Delta)\big)+O(\log r)$, outside I_n, which contradicts Theorem XI. (ii) can be proved similarly as Theorem V by means of (4).

Theorem XIII. Under the same condition as Theorem XII, if $f(z)$-a, where $[a, w_0]<\delta$, has only finite number of zero points in Δ, then there exists a curve Λ ending at a point of E, such that $f(z)\to a$ along Λ.

Proof. Since, by Theorem XII, the Riemann surface F of the inverse function $\varphi(w)$ of $f(z)$ contains infinitely many sheets, a is a boundary point of F, which by Theorem VI, is an accessible boundary point of F. Hence there exists a curve on F ending at a, which corresponds to Λ on the z-plane ending at a point of E, along which $f(z)$ tends to a.