119. On the Newtonian Capacity and the Linear Measure.

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I. Given a bounded set E of points in Euclidean plane ω, let us denote the diameter of E, as usual, by $\delta(E)$. We shall denote, for each $\varepsilon > 0$, by $\wedge_\varepsilon(E)$ the lower bound of all the sums $\sum_i \delta(E_i)$ where $\{E_i\}_{i=1,2,\dots}$ is an arbitrary partition into a sequence of sets that have diameters less than ε and no two of which have common points. Making ε approach to zero, the number $\wedge_\varepsilon$ tends, in a monotone non-decreasing way, to a unique limit (finite or infinite) which is called the linear measure of E and will be denoted by $\wedge(E)$.

It is known that $\wedge(E)$ has the property of Carathéodory's outer measure[1] and therefore all the Borel sets are measurable in the sense of linear measure and $\wedge(E)$ is an additive function of linearly measurable set.

We shall say that μ is a positive distribution of the mass m on the Borel set E, if μ is a non-negative and completely additive set function defined for all the Borel sets in ω such that $\mu(E)=m$ and $\mu(\omega-E)=0$.

Given a fixed point P, a variable point Q, let us denote the distance from P to Q by r_{PQ}. For an arbitrary distribution of positive mass on the set E, the Lebesgue-Stieltjes integral

$$u(p)=\int_E \frac{d\mu(Q)}{r_{PQ}}$$

represents a function ($\leqq+\infty$) of point P which we call the Newtonian potential due to the distribution μ.

For every distribution of the *unit* mass on the set E, the potential $u(p)$, $p\varepsilon E$, has a positive upper bound (finite or infinite). Denoting by $V(E)$ the lower bound of this uppor bound, for all possible distributions, we call

$$C(E)=\frac{1}{V(E)}$$

the newtonian capacity of the set E.

As is known, the Newtonian capacity $C(E)$ is not necessarily addtive even in the restrictive sense.

II. Mr. Frostman has proved in his thesis[2] the following theorem.

Theorem I. If the set E is of linear measure zero, the Newtonian capacity of E is zero.

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1) F. Hausdorff: Dimension und äusseres Mass, Math. ann. Vol. **79** (1919) pp. 157–179.

2) Frostman: Potentiel d'équilibre et capacite des énsemble. Lund (1935) p. 89 Mr. Frostman has proved the theorem concerning more general measure and capacity.

This theorem shows that the linear measure is the measure that is not greater than that of the Newtonian capacity. The theorem we are to prove is as follows:

Theorem. The Borel set E of finite linear measure is of Newtonian capacity zero.

III. We shall summarise here some theorems that will be used in the proof of our theorem.

Theorem 2[1]. The Newtonian capacity of the set E, on each point of which the Newtonian potential due to a certain distribution of finite positive mass is infinite, is equal to zero.

Theorem 3[2]. Let E be the Borel set whose Newtonian capacity is zero and F be an arbitrary Borel set, then $C(E+F)=C(E)+C(F)$.

Denoting the closed circle of radius r with the center P by $S(P, r)$, we call

$$\overline{\lim_{r\to+0}} \frac{\Lambda\big(E\cdot S(P,r)\big)}{2r} = \bar{D}(P, E)$$

the linear upper density of the set E at the point P.

Concerning this upper density Mr. Sierpinski has proved the following theorem.

Theorem 4[3]. For all point P on the set E of finite linear measure, except at most points that belong to the set of linear measure zero, we have

$$\frac{1}{2} \leqq \bar{D}(P, E) \leqq 1 \tag{1}$$

IV. Now we shall proceed to prove our theorem.

We can suppose that $0<\Lambda(E)<+\infty$, for, if $\Lambda(E)$ is zero, it follows $C(E)=0$ from the theorem I. Then we can take Λ as the distribution of finite mass on the set E and therefore consider the Newtonian potential

$$u(P)=\int_E \frac{d\Lambda(Q)}{r_{PQ}}$$

Let P be an arbitrary point whose linear upper density satisfies (I) in theorem 4. From the definition of the integral, we have

$$u(P)=\lim_{r\to0}\int_E \left[\frac{1}{r_{PQ}}\right]_{\frac{1}{r}} d\Lambda(Q) \tag{2}$$

$$\geqq \lim_{r\to0}\int_{E-S(P,r)} \frac{d\Lambda(Q)}{r_{PQ}},$$

1) Frostman: l. c. p. 81.

2) l. c. p. 54.

3) Sierpiński: F. M. **9** (1927). p. 182.

where

$$\left[\frac{1}{r_{PQ}}\right]_{\frac{1}{r}} = \begin{cases} \frac{1}{r_{PQ}} & \text{as } r_{PQ} > r \\ \frac{1}{r} & \text{as } r_{PQ} \leqq r \end{cases}$$

From the hypothesis about the point P and the definition of the upper density, we can find the decreasing sequence $\{r_n\}$ of positive numbers such th

$$\lim_{n\to\infty} r_n = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\Lambda\big(E\cdot S(P, r_n)\big)}{2r_n} = \bar{D}(P, E) \geqq \frac{1}{2}$$

Then, denoting by ε an arbitrary number so that $\frac{\bar{D}(P, E)}{4} > \varepsilon > 0$, there exist a natural number N such that, for all $n > N$, such

$$\frac{\Lambda\big(E\cdot S(P, r_n)\big)}{2r_n} > \bar{D}(P, E) - \varepsilon$$

and

$$\frac{\Lambda\left(E\cdot S\left(P, \frac{r_n}{2}\right)\right)}{r_n} < \bar{D}(P, E) + \varepsilon .$$

For such n, we obtain the following inequality

$$\int_{S(P, r_n) - S\left(P, \frac{r_n}{2}\right)} \frac{d\Lambda(Q)}{r_{PQ}} \geqq \frac{<\big(E\cdot S(P, r_n)\big) - \Lambda\left(E\cdot S\left(P, \frac{r_n}{2}\right)\right)}{r_n}$$

$$> \bar{D}(P, E) - 2\varepsilon > \frac{1}{4} .$$

Hence it follows from (2) and Cauchy's theorem of convergence that

$$\lim_{r\to 0} \int_{E - S(P, r)} \frac{d\Lambda(Q)}{r_{PQ}} = +\infty$$

and

$$u(P) = +\infty .$$

In vertue of theorem 4, we have $u(p) = +\infty$ everywhere on the set F where F is the subset of E such that $\Lambda(E-F) = 0$ and consequently $C(E-F) = 0$. Therefore from theorem 2, it follows $C(F) = 0$, which shows also: by theorem 3

$$C(E) = 0 .$$

V. In the above consideration, we have restricted ourselves to the Borel sets for the sake of simplicity. But the result obtained remains valid also for the generalised definition of capacity due to Frostman. For, let E be any set with $\Lambda(E) < +\infty$. Since Λ is the regular measure[1], there exists a Borel set $H \supset E$ such that $\Lambda(E) = \Lambda(H)$. Then on the one hand, from the result already obtained, we have

1) Saks: Theory of the integral. p. 53.

$C(H)=0$. On the other hand, from the monotony[1] of capacity we have

$$(0 \leqq) C(E) \leqq C(H).$$

Hence we must have

$$C(E)=0.$$

1) Frostman: l. c. p. 49.