117. *Locally Bounded Linear Topological Spaces.*

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D. H. Hyers[1] has introduced the notion of absolute value into locally bounded linear topological spaces, and proved that the absolute value is upper semi-continuous, while J. v. Wehausen[2] showed that a linear topological space is metrizable as an F-metric if and only if it satisfies the first countability axiom. Since every locally bounded linear topological space satisfies the first countability axiom, it is metrizable as an F-metric. But all F-metric spaces are not necessarily locally bounded. Hence the problem arises: what metric spaces are equivalent to locally bounded linear topological spaces?

In this paper we introduce a lower or upper semi-continuous absolute value into locally bounded linear topological space and give a condition that the absolute value is continuous. We define F'-normed spaces and prove that they are equivalent to locally bounded linear topological spaces.

§ 1. *Definitions and lemmas.*

Definition 1. A linear space L is called a linear topological space if there exists a family $\mathfrak{U}$ of sets $U \subset L$ satisfying following conditions[3] ·

1) *The intersection of all the sets* $\in \mathfrak{U}$ *is* $\{\theta\}$.[1)]
2) *If* $U, V \in \mathfrak{U}$ *there exists* $W \in \mathfrak{U}$ *such that* $W \subset U \frown V$.
3) *If* $U \in \mathfrak{U}$ *there exists* $V \in \mathfrak{U}$ *such that* $V+V \subset U$.[2)]
4) *If* $U \in \mathfrak{U}$ *there exists* $V \in \mathfrak{U}$ *such that* $[-1, 1]V \subset U$.[3)]
5) *If* $x \in L$, $U \in \mathfrak{U}$ *there exists real* a *such that* $x \in aU$.

Definition 2. A linear topological space L is called locally bounded if $\mathfrak{U}$ satisfies:

6) *There exists a bounded set*[4)] V *of* $\mathfrak{U}$.

Lemma 1. If we put $H=[-1, 1]V$, *then*

1) $[-1, 1]H=H$.
2) $0<\alpha<\beta$ *implies* $\alpha H \subset \beta H$.
3) H *is bounded.*
4) *For every* $\alpha, \beta \geqq 0$, $\alpha+\beta=1$ *there exists a constant* $k \geqq 1$ *independent of* α, β *such that* $\alpha H+\beta H \subset kH$.
5) *Let* $\mathfrak{U}^*=\{\alpha H\}$, $\alpha>0$. *Then* $\mathfrak{U}^*$ *is equivalent to* $\mathfrak{U}$.

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1) $\{\theta\}$ is the set consisting of zero element θ only.
2) If $S, T \subset L$, $S+T$ is the set of all $x+y$, where $x \in S$, $y \in T$.
3) $[-1, 1]V$ is the set of all αx such as $-1 \leqq \alpha \leqq 1$, $x \in V$.
4) A set S in a linear topological space will be called bounded if for any $U \in \mathfrak{U}$ there is a number α such as $S \subset \alpha U$. (v. Neumann) This is the same to say that for and sequence $\{x_n\} \subset S$ and any real sequence $\{\alpha_n\}$ converging to 0, the sequence $\{\alpha_n x_n\}$ converges to θ. (Banach and Kolmogoroff)

Proof. 1), 2) and 3) are clear. We begin by proving 3). Given $U \in \mathfrak{U}$, there exists $W \in \mathfrak{U}$ such that $[-1, 1]W \subset U$. Since V is bounded, there exists a real α satisfying $V \subset \alpha W$. Hence $H = [1, 1]V \subset \alpha[-1, 1]W \subset \alpha U$. To prove 4), suppose that such k does not exist. Then there exists a sequence $\{\alpha_n x_n + \beta_n y_n\}$ such that $\alpha_n, \beta_n \geqq 0$, $\alpha_n + \beta_n = 1$, $x_n, y_n \in H$ and $\alpha_n x_n + \beta_n y_n \bar{\in} nH$ $(n = 1, 2, \ldots)$. Therefore $(\alpha_n/n)x_n + (\beta_n/n)y_n \bar{\in} H$ $(n = 1, 2, \ldots)$ and $\alpha_n/n, \beta_n/n \to 0$. From boundedness of H and the continuity of addition we have $(\alpha_n/n)x_n + (\beta_n/n)y_n \to \theta$. This is a contradiction. Now if $k < 1$, $H \subset kH \subset k^2H \subset \cdots$. By the boundedness of H it follows $H = \{\theta\} = V$. This contradicts Definition 1. 5).

In the following line we assume k as a fixed number satisfying 4).

Lemma 2. If we put $G = [-1, 1]V_i$[5], then G is an open[5] *set satisfying 1)–5) in Lemma 1.*

Proof. 1)–5) is proved analogously as Lemma 1.

We will first show that $\alpha V_i (\alpha \neq 0)$ is open. Since V_i is open, for every $x \in V_i$ there exists $U \in \mathfrak{U}$ such that $x + U \subset V_i$. So that $\alpha x + \alpha U \subset \alpha V_i$. There is a $W \in \mathfrak{U}$ such as $\frac{1}{\alpha} W \subset U$. Hence $\alpha x + W \subset \alpha x + \alpha U \subset \alpha V_i$. Thus αV_i is open. Now

$$G = \bigvee_{|\alpha| \leq 1} \alpha V_i = \{\bigvee_{\substack{|\alpha| \leq 1 \\ \alpha \neq 0}} \alpha V_i\} \cup \{\theta\} = \bigvee_{\substack{|\alpha| \leq 1 \\ \alpha \neq 0}} \alpha W_i .$$

Therefore G is open.

Lemma 3. If we put $F = [-1, 1]V_{cl}$[6]*, then F is closed*[7] *satisfying 1)–5) in Lemma 1.*

Proof. From the definition of F, 1) and 2) are obvious. In order to prove 3), 4) and 5) it suffices to show that V_{cl} is bounded. For a given $U \in \mathfrak{U}$, there exists W such that $W + W \subset U$. By the boundedness of V there exists a real α such as $V \subset \alpha W$, and then $V_{cl} \subset V + V \subset \alpha(W + W) \subset \alpha U$.

It remains to prove F is closed. Let $y \in F_{cl}$. Since L satisfies the first countability axiom, there exists a sequence $\{\alpha_n x_n\} \subset F$ such that $y_n = \alpha_n x_n \to y$, $\alpha_n \in [-1, 1]$, $x_n \in V_{cl}$. Let α be a limiting point of $\{\alpha_n\}$, then there exists a subsequence $\{\alpha_{n_k}\}$ such that $\alpha_{n_k} \to \alpha \in [-1, 1]$. If $\alpha = 0$, $y = \theta \in F$, because F is bounded. If $\alpha \neq 0$ then $\lim x_{n_h} = \lim (1/\alpha_{n_k}) y_{n_k} = (1/\alpha)y$. Let $\lim x_{n_k} = x$, then $x \in V_{cl}$, thus $y = \alpha x \in F$. This completes the proof.

§ 2. *Absolute value.*

Definition 3. A linear space L will be said to be an absolute valued space, if corresponding to each $x \in L$ there is a real number $|x|$ (it is called absolute value of x) with the properties:

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5) In the linear topological space L, $S \subset L$ is called open if $S = S_i$. S_i is the set of all x for which there exists a $U \in \mathfrak{U}$ with $x + U \subset S$.

6) V_{cl}, the closure of V, is defined by $V_{cl} = C((CV)_i)$. (C denotes complementation).

7) F is defined as closed if $F = F_{cl}$

1) $|x| \geqq 0$; $|x|=0$ *implies* $x=\theta$.

2) $|ax|=|a||x|$ *for every real number* a.

3) $|x+y| \leqq k|x|+k|y|$ *where* $k \geqq 1$, *and independent to* x, y.

Theorem 1. In every locally bounded linear topological space L it can be introduced an absolute value which is equivalent to the original topology.

Conversely, in every absolute valued space L, if we define the fundamental system by

$$\mathfrak{U}'=\{aS'\},\ a>0 \text{ where } S'=(x:|x|<1)$$

or

$$\mathfrak{U}''=\{aS''\},\ a>0 \text{ where } S''=(x:|x|\leqq 1),$$

L becomes a locally bounded linear topological space.

Proof. Let $|x|=\underset{x\in aH}{\text{g.l.b.}}|a|$. 1), 2) and equivalency of $\mathfrak{U}$ and the absolute value have been proved by D. H. Hyers[1].

We will show 3). Let $|x|=\alpha$, $|y|=\beta$. For every $\varepsilon>0$ $x\in(\alpha+\varepsilon)H$, $y\in(\beta+\varepsilon)H$. From Lemma 1. 4) it is easily seen that

$$x+y\in(\alpha+\varepsilon)H+(\beta+\varepsilon)H\subset k(\alpha+\beta+2\varepsilon)H,$$

i. e.

$$|x+y|\leqq k(\alpha+\beta)=k|x|+k|y|.$$

The converse is clear.

Corollary. If $k=1$, the absolute value is norm, i. e. $|x|$ satisfies triangular inequality.

Remarks. In this case H is convex.

$\mathfrak{U}'$ is equivalent to $\mathfrak{U}''$. And let

$$|x|'=\underset{x\in aS'}{\text{g.l.b.}}|\alpha|, \qquad |x|''=\underset{x\in aS''}{\text{g.l.b.}}|\beta|.$$

Then $|x|'=|x|''$.

Example 1. Let $l^{1/p}(p\geqq 1)$ be the set of sequence of real numbers $x=\{x_n\}$ such that $\sum_{n=1}^{\infty}|x_n|^{1/p}<\infty$. We define $|x|=(\sum_{n=1}^{\infty}|x_n|^{1/p})^p$. Then $l^{1/p}$ is an absolute valued space with $k=2^{p-1}$. Since

$$|x+y|=(\sum_{n=1}^{\infty}|x_n+y_n|^{1/p})^p\leqq(\sum_{n=1}^{\infty}|x_n|^{1/p}+\sum_{n=1}^{\infty}|y_n|^{1/p})^p$$
$$\leqq 2^{p-1}\Big((\sum_{n=1}^{\infty}|x_n|^{1/p})^p+(\sum_{n=1}^{\infty}|y_n|^{1/p})^p\Big)=2^{p-1}(|x|+|y|).$$

Example 2. Let $L^{1/p}(p\geqq 1)$ denote the set of measurable functions $x(t)$ defined on $E=(0,1)$ and such that $\int_E|x(t)|^{1/p}dt<\infty$. If the absolute value of x is defined by $|x|=\Big(\int_E|x(t)|^{1/p}dt\Big)^p$, then $L^{1/p}$ is an absolute valved space with $k=2^{p-1}$.

Theorem 2. Let $|x|=\underset{x\in aG}{\text{g.l.b.}}|a|$, *then* $|x|$ *is an upper semi-continuous absolute valve, i. e.* $|x|$ *satisfies* 1), 2), 3) *of Definition 3 and*

4) *for every* $x\in L$ *and* $\varepsilon>0$ *there exists* $\delta>0$ *such that* $|y|-|x|<\varepsilon$ *for* $|y-x|<\delta$.

Conversely for every linear space with an upper semi-continuous absolute value we can take as the fundamental system a family of all open sets.

Proof. The first half has been proved by D. H. Hyers.[1]

To prove the converse it is sufficient to show that $S'=(x:|x|<1)$ is an open set with respect to $\mathfrak{U}'$-topology. Let $x\in S'$, then $|x|<1$. Take $\varepsilon<0$ such that $|x|+\varepsilon<1$. Since the absolute value is upper semi-continuous at x, there exists $\delta>0$ satisfying $|y|-|x|<\varepsilon$ for $|y-x|<\delta$. We have $|y|<|x|+\varepsilon<1$ for $y\in x+\delta S'$. Thus S' is open.

Theorem 3. Let $|x|=\underset{x\in aF}{\text{g. l. b.}}|\alpha|$, *then* $|x|$ *is a lower semi-continuous absolute value, i. e.* $|x|$ *satisfies* 1), 2), 3) *of Definition 3 and*

5) *For every* $x\in L$ *and* $\varepsilon>0$ *there exists* $\delta>0$ *such that* $|x|-|y|<\varepsilon$ *fcr* $|x-y|<\delta$.

Conversely, for every linear space with a lower semi-continuous absolute value we can take as the fundamental system a family of all closed sets.

Proof. 1), 2), 3) is clear. We will show 5). Since F is closed, $aF(\alpha>0)$ is closed. Since $|x|$ is continuous at θ, it is sufficient to prove that $|x|$ is lower semi-continuous at $x\neq\theta$. Take $\varepsilon>0$ satisfying $|x|-\varepsilon>0$. It follows $x\bar{\in}(|x|-\varepsilon)F$. Then there exists $\delta>0$ such that $y\bar{\in}(|x|-\varepsilon)F$ for $|y-x|<\delta$, or $|y|>|x|-\varepsilon$ for $|y-x|<\delta$. Thus $|x|-|y|<\varepsilon$ for $|y-x|<\delta$.

To prove the converse it is sufficient to show that $S''=(x:|x|\leqq 1)$ is closed with respect to $\mathfrak{U}''$-topology. $x\bar{\in}S''$, then $|x|>1$. If we take ε such as $|x|-\varepsilon=1$, $\varepsilon>0$, then by the lower semi-continuity of $|x|$, there exists $\delta>0$ such that $|x|-|y|<\varepsilon$ for $|x-y|<\delta$. That is $y\bar{\in}(|x|-\varepsilon)S''=S''$ for $y\in x+\delta S''$. Thus for $x\bar{\in}S''$ there exists a neighbourhood of x which does not intersect S''. This shows that S'' is closed.

We notice that for a given locally bounded linear topological space the absolute values introduced in Theorem 1, 2, 3 are not equivalent each other in general.

Example 3. Let L be the linear space of complex numbers $x=\alpha+\beta i$ (α,β real) with absolute value

$$|x|=\begin{cases}(\alpha^2+\beta^2)^{1/2} & \text{for } \beta\neq 0,\\ \frac{1}{2}|\alpha| & \text{for } \beta=0.\end{cases}$$

Then L is a linear space with upper semi-continuous absolute value and $k=\sqrt{8}$. $\mathfrak{U}'$ and $\mathfrak{U}''$ are equivalent to usual topology.[1]

Example 4. Let L be the above linear space which has the following absolute value

$$|x|=\begin{cases}(\alpha^2+\beta^2)^{1/2} & \text{for } \beta\neq 0,\\ 2|\alpha & \text{for } \beta=0.\end{cases}$$

In this case L is a linear space which has lower semi-continuous absolute value with $k=\sqrt{2}$, and $\mathfrak{U}', \mathfrak{U}''$ are also equivalent to usual topology.

Above examples show that there exist locally bounded linear topological spaces for which the absolute values are not continuous. When $|x|$ is continuous? Answer is given by the following theorem.

Theorem 4. The absolute value of a given absolute valued space L is continuous if and only if $S'=(x:|x|<1)$ is open, and $S''=(x:|x|\leqq 1)$ is closed.

Proof. Suppose that $|x|$ is continuous. By Theorem 2, S' is open with respect to $\mathfrak{U}'$-topology. By Theorem 3, S'' is closed with respect to $\mathfrak{U}''$-topology. Since $\mathfrak{U}'$ is equivalent to $\mathfrak{U}''$, and S' is open, S'' is closed with respect to both topologies. Conversely, by the assumption $\alpha S'(\alpha>0)$ is open and $\alpha S''(\alpha>0)$ is closed. From Theorem 1, Remark and Theoreme 2, 3, $|x|$ is continuous.

In Exampes 2, 3, S' are open and S'' are closed. Therefore both absolute values are continuous. In Example 3, S' is open, but S'' is not closed. In Example 4, S'' is closed, but S' is not open. It follows that these absolute values are not continuous.

§ 3. *F'-norm.*

It is convenient to introduce the following definition.

Definition 4. A linear space L will be called an F'-normed space if for any $x\in L$ there corresponds a real number $\|x\|$, called F'-norm, with following properties:

1) $\|x\|\geqq 0$: $\|x\|=0$ *implies* $x=\theta$.
2) *For any real α and $x\in L$, there is a real $|\tilde{\alpha}|$ such that $\|\alpha x\|=|\alpha|\,\|x\|$. In fact $|\tilde{\alpha}|=C^r$ then $|\tilde{\alpha}|=2^r$, where C is a fixed constant independ of α and x.*
3) $\|x+y\|\leqq\|x\|+\|y\|$.

Remark. If we define $(x, y)=\|x-y\|$ for an F'-normed space we have an F-metric space. So that F'-norm is a special case of F'-metric.

Theorem 5. An F'-norm which is equivalent to the original topology can be introduced into every locally bounded linear topological spece, and conversely every F'-normed space is a locally bounded linear topological space.

The proof of the Theorem will be devided into four steps.

(1) For F in Lemma 3, there exists a $k'\geqq 3$ such that $F+F+F\subset k'F$. Take $k'=C$ in Definition 4, 2) and let $|x|'=\underset{x\in\alpha F}{\text{g. l. b.}}\,|\tilde{\alpha}|$. Then $|x|'$ has following properties:

1) $|x|'\geqq 0$; $|x|'=0$ implies $x=\theta$.
2) $|\alpha x|'=|\tilde{\alpha}|\,|x|$ where $C=k'$.
3) $|x|', |y|', |z|'\leqq\delta$ imply $|x+y+z|'\leqq 2\delta$.
4) $|x|'$ is equivalent to the original topology.

If we notice that $|x|'=|\tilde{x}|$ for $|x|=\underset{x\in\alpha F}{\text{g. l. b.}}\,|\alpha|$, then 1), 2) 4) are obvious. It remains to prove 3). Let $\delta=2^r$. $|x|', |y|', |z|', \leqq\delta$ implies

$x, y, z \in k'^{\gamma}F$, and then $x+y+z \in k'^{\gamma}(F+F+F) \subset k'^{\gamma+1}F$. Thus $|x+y+z|' \leqq 2^{\gamma+1}=2\delta$.

(2) Let us put $\|x\| = \text{g. l. b.}_{x_0=\theta,\, x_n=x} \sum_{i=1}^{n} \|x_i - x_{i-1}|'$, where $\{x_1, x_2, \dots, x_{n-1}\} \subset L$ and n are arbitrary. Then $\|x\|$ satisfies 1), 2), 3) of Definition 4 with $C=k'$.

1) is clear. We will show 2). For a given $\varepsilon > 0$, take $\delta > 0$ such that $\delta|\alpha| < \varepsilon$. From the definition of $\|x\|$, there exists a sequence $\{x_0=\theta,\ x_1, x_2, \dots, x_n = x\}$ satisfying $\|x\|+\delta > \sum_{i=1}^{n} |x_i - x_{i-1}|'$. Thus follows

$$|\tilde{\alpha}|\,\|x\|+\varepsilon > |\tilde{\alpha}|\,\|x\|+|\tilde{\alpha}|\,\delta > \sum_{i=1}^{n} |\alpha x_i - \alpha x_{i-1}|' \geqq \|\alpha x\|.$$

That is $|\tilde{\alpha}|\,\|x\| \geqq \|\alpha x\|$. In order to prove the converse inequality, if we notice that $\alpha=0$ if and only if $|\tilde{\alpha}|=0$, then we can assume that $\alpha \neq 0$. For given $\varepsilon > 0$, there exists a sequence $\{y_0=\theta,\ y_1, y_2, \dots, y_n = \alpha x\}$ with $\|\alpha x\|+\varepsilon > \sum_{i=1}^{n} |y_i - y_{i-1}|'$. It follows

$$\sum_{i=1}^{n} |y_i - y_{i-1}|' = |\tilde{\alpha}| \sum_{i=1}^{n} \left| \frac{1}{\alpha} y_i - \frac{1}{\alpha} y_{i-1} \right|' \geqq |\tilde{\alpha}|\,\|x\|.$$

Thus we have $\|\alpha x\| \geqq |\tilde{\alpha}|\,\|x\|$.

We will now prove the triangular inequality of $\|x\|$. For a given $\varepsilon > 0$ there exist $\{x_0, x_1, \dots, x_n\}$ and $\{y_0, y_1, \dots, y_m\}$ such that

$$\|x\|+\frac{\varepsilon}{2} > \sum_{i=1}^{n} |x_i - x_{i-1}|', \qquad \|y\|+\frac{\varepsilon}{2} > \sum_{i=1}^{m} |y_i - y_{i-1}|'.$$

If we put

$$Z_i = \begin{cases} x_i & \text{for } \ 0 \leqq i \leqq n, \\ x+y_i & \text{for } \ i > n, \end{cases}$$

then

$$\|x\|+\|y\|+\varepsilon > \sum_{i=1}^{n+m} |z_i - z_{i-1}|' \geqq \|x+y\|.$$

Thus we obtain $\|x+y\| \leqq \|x\|+\|y\|$.

(3) $\|x\|$ is equivalent to $|x|'$.

For, following Birkhoff's argument[2] we can prove the inquality $\frac{1}{2}|x|' \leqq \|x\| \leqq |x|'$.

(4) Every F'-normed space is a locally bounded linear topological space, i. e. if we take $\mathfrak{U}' = \{S_\alpha\}$ where $S_\alpha = (x\colon \|x\| < \alpha, \alpha > 0)$ the fundamental system, then $\mathfrak{U}'$ satisfies 1)–6) in Definition 1 and 2.

The proof is easy.

From Theorems 1 and 5 we obtain the following.

Theorem 6. Locally bounded linear topological space, absolute valued spaces and F'-normed spaces are topologically equivalent to each other.

§ 4. *Complex linear topological spaces.*

In the above theory we can replace the real operator domain by complex number field. That is, complex linear topological space is defined by Definition 1 as space with complex number field as operator domain, with modification of 4) such as

4′) Given $U \varepsilon \mathfrak{U}$ there exists $V \varepsilon \mathfrak{U}$ such as IV $\subset \mathfrak{U}$, where $I=(\alpha: |\alpha| \leqq 1$, α complex number).

Let $H=IV$, $G=IV_i$ and $F=IV_{cl}$. Then without any formal change we can use Definition 3 and 4. Lemmas, theorems and corrolaries are all valid in this case.

References.

[1] D. H. Hyers, Revista de Ciencias Lima, Vol. **41** (1939), pp. 555–574.

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[3] J. von Neumann, Trans. Amer. Math. Soc., Vol. **37** (1935), pp. 1–20. We shut out the trivial case, i. e. L consists of ero element only.