47. On the Domain of Existence of an Implicit Function defined by an Integral Relation $G(x, y)=0$.

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1. *Theorems of Julia and Gross.*

Let $G(x, y)$ be an integral function with respect to x and y and $y(x)$ be an analytic function defined by $G(x, y)=0$ and F be its Riemann surface spread over the x-plane. Let E be a set of points on the x-plane, which are not covered by F. Evidently E is a closed set.

Julia[1] proved that E does not contain a continuum. If $y(x)$ is an algebroid function of order n, such that $A_0(x)y^n+A_1(x)y^{n+1}+\cdots+A_n(x)=0$, where $A_i(x)$ are integral functions of x, then F consists of n sheets and covers every point on the x-plane exactly n-times, where a branch point of F of order k is considered as covered k-times by F. We will prove

Theorem I. If $y(x)$ is not an algebroid function of x, then F covers any point on the x-plane infinitely many times, except a set of points of capacity zero.

In this paper "capacity" means "logarithmic capacity."

If we interchange x and y, we have

Let $G(x, y)$ be an integral function with respect to x and y and $y(x)$ be an analytic function defined by $G(x, y)=0$. If $y(x)$ does not satisfy a relation of the form: $A_0(y)x^n+A_1(y)x^{n+1}+\cdots+A_n(y)=0$, where $A_i(y)$ are integral functions of y, then $y(x)$ takes any value infinitely many times, except a set of values of capacity zero.

This is a generalization of Picard's theorem for a transcendental meromorphic function for $|x|<\infty$.

Julia's proof depends on the following

Gross' theorem[2]: Let $f(z)$ be one-valued and regular on the Riemann surface F, which does not cover a continuum. If $f(z)$ tends to zero, when z tends to any accessible boundary point of F, then $f(z)\equiv 0$.

We will first extend this Gross' theorem in the following way.

Theorem II. Let $f(z)$ be one-valued and meromorphic on a connected piece F of its Riemann surface, whose projection on the z-plane lies inside a Jordan curve C and F do not cover a closed set E of positive capacity, which lies with its boundary entirely inside C. If $f(z)$ tends to zero, when z tends to any accessible boundary point of F, whose projection on the z-plane lies inside C, except enumerably infinite number of such accessible boundary points, then $f(z)\equiv 0$.

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1) G. Julia: Sur le domaine d'existence d'une fonction implicite défine par une relation entière $G(x, y)=0$. Bull. Soc. Math. (1926).

2) W. Gross: Zur Theorie der Differentialgleichungen mit festen kritischen Punkten. Math. Ann. **78** (1918).

2. Priwaloff's theorem.

We use the following Priwaloff's theorem[1] in the proof.

Theorem III. Let $f(z)$ be meromorphic in $|z|<1$ and E be a measurable set of positive measure on $|z|=1$. If $f(z)$ tends to zero, when z tends to any point of E by the curves non-tangential to $|z|=1$, then $f(z)\equiv 0$.

I will give a simple proof for the sake of completeness.

Proof. We map $|z|<1$ on $\Re(s)>0$ on the $s=\sigma+it=re^{i\theta}$-plane by $z=\varphi(s)$ and put $F(s)=f\big(\varphi(s)\big)$. Then E corresponds to a set e of positive measure on the t-axis. Let $\varDelta_{r_0}$ be a triangle determined by three points: $0, r_0e^{i\theta_0}, r_0e^{-i\theta_0}$ $\left(0<\theta_0<\frac{\pi}{2}\right)$ and s_n $(n=1,2,\dots)$ be rational points in $\varDelta_{r_0}$, whose coordinates are rational numbers. We put $F_n(t)=|F(s_n+it)|$ and

$$\varPhi_{r_0}(t)=\text{upper}\lim_{n}\text{it}\, F_n(t)\,. \tag{1}$$

Then
$$\varPhi_{r_0}(t)=\underset{s\in\varDelta_{r_0}}{\text{upper limit}}\,|F(s+it)|\,. \tag{2}$$

Since $F_n(t)$ is continuous, $\varPhi_{r_0}(t)$ is a measurable function and by the hypothesis, $\lim_{r_0\to 0}\varPhi_{r_0}(t)=0$ on e. Hence by Egoroff's theorem, $\lim_{r_0\to 0}\varPhi_{r_0}(t)=0$ uniformly on a bounded closed sub-set e_1 of e, such that $me_1>0$.

Hence from (2) we have for a small r_0,

$$|F(s+it)|<\varepsilon\,,\quad \text{for}\quad s\in\varDelta_{r_0}\,,\quad t\in e_1\,. \tag{3}$$

Let $\varDelta(t)$ be a triangle determined by three points: it, $it+r_0e^{i\theta_0}$, $it+r_0e^{-i\theta_0}$. We add all such triangles for $t\in e_1$ and put $\varDelta_1=\sum_{t\in e_1}\varDelta(t)$. Let $\varDelta_2$ be a rectangle: $r_0\cos\theta_0\leqq\sigma\leqq R_0$, $|t|\leqq M$, such that $F(s)$ has no poles on the boundary of $\varDelta_2$. We put $\varDelta=\varDelta_1+\varDelta_2$, then the boundary $\varGamma$ of $\varDelta$ is a rectifiable curve, which meets the t-axis in e_1 and $F(s)$ is bounded in the neighbourhood of $\varGamma$ and tends to zero, when s tends to e_1 from the inside of $\varGamma$. If we consider e_1 as a set on $\varGamma$, then its measure defined by arc length of $\varGamma$ is positive. Hence if we map the inside of $\varGamma$ on $|\zeta|<1$ by $s=\psi(\zeta)$, then, by F. and M. Riesz' theorem[2], e_1 corresponds to a set ϵ_1 of positive measure on $|\zeta|=1$. Let $G(\zeta)=F\big(\psi(\zeta)\big)$ and $\zeta_1,\zeta_2,\dots,\zeta_n$ be the poles of $G(\zeta)$ in $|\zeta|<1$ and $H(\zeta)=G(\zeta)\prod_{\nu=1}^{n}\frac{\zeta-\zeta_\nu}{1-\bar{\zeta}_\nu\zeta}$, then $H(\zeta)$ is regular and bounded in $|\zeta|<1$ and tends to zero, when ζ tends to any point of ϵ_1. Hence by the well known theorem, $H(\zeta)\equiv 0$, or $f(z)\equiv 0$, q. e. d.

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1) M. J. Priwaloff: Sur certaines propriétes métriques des fonctions analytiques. Jour. d. l'ecole polytechnique. (1924).

2) F. u. M. Riesz: Über die Randwerte analytischer Functionen. 4. congr. scand. math. Stockholm. 1916.

3. *Proof of Theorem II.*

Let $\mathfrak{F}$ be the simply connected universal covering Riemann surface of F. We map $\mathfrak{F}$ on $|x|<1$ by $z=\varphi(x)$ and put $F(x)=f(\varphi(x))$. Since $\varphi(x)$ is bounded in $|x|<1$, by Fatou's theorem, $\lim \varphi(x)$ exists almost everywhere on $|x|=1$, when x tends to $|x|=1$ non-tangentially.

Let $u(z)$ be the solution of the Dirichlet problem for the schlicht domain bounded by C and E with the boundary condition that $u(z)=0$ on C and $u(z)=1$ on the boundary of E. Then, since cap. $E>0$ we have $u(z)\not\equiv 0$. If by the mapping $z=\varphi(x)$, C corresponds to a set of measure 2π on $|x|=1$, then any bounded harmonic function on $\mathfrak{F}$, which vanishes on the points of $\mathfrak{F}$ above C, would vanish identically. But the above solution $u(z)$ of the Dirichlet problem, considered as a bounded harmonic function on $\mathfrak{F}$, vanishes on the points of $\mathfrak{F}$ above C and does not vanish identically. Hence C corresponds to a set of measure $<2\pi$ on $|x|=1$, so that the accessible boundary points of F, whose projections on the z-plane lies inside C correspond to a set e_1 of positive measure on $|x|=1$. Since, by F. Riesz' theorem, the set on $|x|=1$, which corresponds to a given point, is of measure zero, the exceptional accessible boundary points in the Theorem correspond to a set e_0 of measure zero on $|x|=1$. Hence if we put $e=e_1-e_0$, then $me=me_1>0$. By the hypothesis, $F(x)$ tends to zero, when x tends to any point of e non-tangentially to $|x|=1$. Hence by Theorem III, $F(x)\equiv 0$, or $f(z)\equiv 0$, q. e. d.

4. *Proof of Theorem I.*

First we will prove a lemma.

Lemma. If a disc K_0 is covered exactly n-times by F, then $y(x)$ becomes an algebroid function of order n.

Proof. Let G be a connected domain containing K_0, such that every point of G is a center of a disc, which is covered exactly n-times by F and E be its boundary. We will prove that G coincides with the finite plane $|x|<\infty$. Suppose that E contains points in the finite distance. From the definition of G, every point x_0 ($\neq\infty$) on E is covered at most n-times by F. If x_0 is covered n-times by F, then the part of F above a small disc K about x_0 contains n discs: $F_1,\dots,F_n$ consisting of only inner points of F, where a piece of the Riemann surface of $(x-x_0)^{\frac{1}{k}}$ above K is considered as k discs.

If there is no connected piece of F above K other than $F_1,\dots,F_n$, then K is covered exactly n-times by F, so that K belongs to G, which contradicts the hypothesis, that x_0 is a boundary point of G. Hence there is another connected piece F_0 of F above K other than $F_1,\dots,F_n$. Then F_0 does not cover the common part G_0 of G and K from the definition of G. Since, as Julia proved, $\frac{1}{y(x)}$ tends to zero, when x tends to any accessible boundary point of F and cap. $G_0>0$, if we apply Theorem II to F_0, we would have $\frac{1}{y(x)}\equiv 0$, which is absurd. Hence every point of E is covered at most $(n-1)$-times by F. Let E_k be a sub-set of E, such that every point of E_k is covered at most k-times by F, then E_k is a closed set and $E_0 \subset E_1 \subset \cdots \subset E_{n-1} = E$. We will prove that cap. $E=0$.

Suppose that cap. $E>0$, then there is a certain k $(0 \leqq k \leqq n-1)$, such that

$$\text{cap.}\, E_0=0,\ \text{cap.}\, E_1=0,\ \ldots,\ \text{cap.}\, E_{k-1}=0,\ \text{cap.}\, E_k>0. \qquad (4)$$

We put $E_k^0=E_k-E_{k-1}$, then cap. $E_k^0=$cap. $E_k>0$. Let e_k^0 be a closed sub-set of E_k^0, such that cap. $e_k^0>0$. Then there exists a point x_0 on e_k^0, such that cap. $e_k^0(K)>0$, a fortiori, cap. $E_k(K)>0$ for any small disc K about x_0, where we denote the part of a set e inside a disc K by $e(K)$.

Since $x_0 \in E_k^0$, x_0 is covered k-times by F. Hence the part of F above a small disc K about x_0 contains k discs: $F_1, \ldots, F_k$ consisting of only inner points of F. Since $k \leqq n-1$, there is another connected piece F_0 of F above K other than $F_1, \ldots, F_k$. Since $E_k(K_0)$ is covered k-times in $F_1, \ldots, F_k$ by F, from the definition of E_k, F_0 does not cover $E_k(K_0)$, where K_0 is a disc about x_0 contained in K.

Since $\dfrac{1}{y(x)}$ tends to zero, when x tends to any accessible boundary point of F, and cap. $E_k(K_0)>0$, if we apply Theorem II to F_0, we would have $\dfrac{1}{y(x)} \equiv 0$, which is absurd. Hence cap. $E=0$, so that every point of E is an accessible boundary point.

Let $y_1(x), \ldots, y_n(x)$ be n branches of $y(x)$ outside E and x_0 $(\neq \infty)$ be any point of E. Suppose that $\dfrac{1}{y_1(x)}, \ldots, \dfrac{1}{y_s(x)}$ have essential singularities and $\dfrac{1}{y_{s+1}(x)}, \ldots, \dfrac{1}{y_n(x)}$ have algebraic singularities at x_0. We put

$$\left.\begin{aligned} \prod_{i=1}^{s}\left(\frac{1}{y}-\frac{1}{y_i(x)}\right) &= \frac{1}{y^s}+\frac{a_1(x)}{y^{s-1}}+\cdots+a_s(x), \\ \prod_{i=s+1}^{n}\left(\frac{1}{y}-\frac{1}{y_i(x)}\right) &= \frac{1}{y^{n-s}}+\frac{b_1(x)}{y^{n-s-1}}+\cdots+b_{n-s}(x), \end{aligned}\right\} \qquad (5)$$

then $a_i(x)$ are one-valued and meromorphic outside E and since $\dfrac{1}{y_i(x)}$ $(i=1, 2, \ldots, s)$ tends to zero, when x tends to any point of E in the neighbourhood U of x_0, $a_i(x)$ are bounded in U, so that, since cap. $E=0$, $a_i(x)$ are regular at x_0[1]. Since $b_i(x)$ are meromorphic at x_0, if we put

$$\prod_{i=1}^{n}\left(\frac{1}{y}-\frac{1}{y_i(x)}\right)=\frac{1}{y^n}+\frac{c_1(x)}{y^{n-1}}+\cdots+c_n(x), \qquad (6)$$

then $c_i(x)$ are meromorphic at x_0, so that the neighbourhood of x_0 is covered exactly n-times by F, which contradicts the hypothesis, that x_0 is a boundary point of G. Hence G coincides with the finite plane

1) R. Nevanlinna: Eindeutige analytische Funktionen. p. 132.

$|x|<\infty$. Then $c_i(x)$ are meromorphic functions for $|x|<\infty$. Consequently $y(x)$ satisfies a relation of the form: $A_0(x)y^n+A_1(x)y^{n-1}+\cdots+A_n(x)=0$, where $A_i(x)$ are integral functions of x. Thus the lemma is completely proved.

By this lemma, we can prove Theorem I simply as follows.

Suppose that $y(x)$ is not an algebroid function and its Riemann surface F does not cover a set E of positive capacity infinitely many times. Let E_k be a set of points, which are covered at most k-times by F. Then E_k is a closed set and $E_0\subset E_1\subset\cdots\subset E_k\subset\cdots, E=\sum_{k=0}^{\infty}E_k$.

Since cap. $E>0$, there is a certain k $(0\leqq k<\infty)$, such that

$$\text{cap.}\,E_0=0,\ \text{cap.}\,E_1=0,\ \ldots,\ \text{cap.}\,E_{k-1}=0,\ \text{cap.}\,E_k>0. \tag{7}$$

Let $E_k^0=E_k-E_{k-1}$, then cap. $E_k^0=$cap. $E_k>0$ and e_k^0 be a closed sub-set of E_k^0, such that cap. $e_k^0>0$. Then there exists a point x_0 on e_k^0, such that cap. $e_k^0(K)>0$, a fortiori, cap. $E_k(K)>0$ for any small disc K about x_0. Since $x_0\in E_k^0$, x_0 is covered k-times by F, hence the part of F above a small disc K about x_0 contains k discs: $F_1,\ldots,F_k$ consisting of only inner points of F. Since $y(x)$ is not an algebroid function, we see by the lemma, that there is another connected piece F_0 of F above K other than $F_1,\ldots,F_k$. Since $E_k(K_0)$ is covered k-times in $F_1,\ldots,F_k$ by F, from the definition of E_k, F_0 does not cover $E_k(K_0)$, where K_0 is a disc about x_0 contained in K. Since $\dfrac{1}{y(x)}$ tends to zero, when x tends to any accessible boundary point of F and cap. $E_k(K_0)>0$, if we apply Theorem II to F_0, we would have $\dfrac{1}{y(x)}\equiv 0$, which is absurd. Hence cap. $E=0$, q. e. d.

5. *Extension of Iversen's theorem.*

We will prove the following extension of Iversen's theorem[1].

Theorem IV. Let $G(x,y)$ be an integral function with respect to x and y and $y(x)$ be an analytic function defined by $G(x,y)=0$ and F be its Riemann surface spread over the x-plane and suppose that $y(x)$ is not an algebroid function of x. If x_0 $(\neq\infty)$ is covered finite times by F, then x_0 is an asymptotic value of the inverse function $x=x(y)$ of $y=y(x)$.

Proof. Let x_0 be covered k-times by F. We denote the disc: $|x-x_0|\leqq\dfrac{\delta}{2^n}$ by K_n $(n=0,1,2,\ldots)$. Then for a small δ, the part of F above K_0 contains k discs: $F_0^{(1)},\ldots,F_0^{(k)}$ consisting of only inner points of F. Since $y(x)$ is not an algebroid function, we see from the lemma, that there is another connected piece F_0 of F above K_0 other than $F_0^{(1)},\ldots,F_0^{(k)}$. Since x_0 is covered k-times in $F_0^{(1)},\ldots,F_0^{(k)}$ by F, F_0 does not cover x_0. Let E_0 be a set of points in K_0 which are not covered by F_0, then as we have proved in § 4, cap. $E_0=0$. Hence there is a point ξ_0 in K_2, which is covered by F_0. Let (ξ_0) be such a

1) F. Iversen: Recherches sur les fonctions inverses des fonctions meromorphes. Thèse. Helsingfors. 1914.

point on F_0 above ξ_0, where we denote a point on F, whose projection on the x-plane is x by (x).

Let F_1 be the connected part of F_0 above K_1, which contains (ξ_0). Similarly we see that there exists a point (ξ_1) on F_1, whose projection ξ_1 lies inside K_3. We connect (ξ_0) and (ξ_1) by a curve (L_0) on F_0, whose projection on the x-plane we denote by L_0. By the similar way, we have points (ξ_n) and curves (L_n) on a connected piece F_n $(F_0 \supset F_1 \supset \cdots \supset F_n)$ above K_n, such that ξ_n lies in K_{n+2} and L_n lies in K_n, so that $\xi_n \rightarrow x_0$, $L_n \rightarrow x_0$. Hence if we put $L = \sum_{n=0}^{\infty} L_n$, then L is a continuous curve on the x-plane tending to x_0. To L, there corresponds on the y-plane, a curve tending to infinity. Hence x_0 is an asymptotic value of the inverse function $x = x(y)$ of $y = y(x)$, q. e. d.

6. *Direct transcendental singularities.*

Let (x_0) be a boundary point of the Riemann surface F of $y(x)$. Iversen called (x_0) a direct transcendental singularity of $y(x)$, if x_0 is lacunary for a connected piece F_0 of F above a certain disc K about x_0, which contains (x_0) as its boundary. We will prove that the set of points on the x-plane, which are the projections of direct transcendental singularities is of capacity zero.

In § 4 we have proved that the set e in a disc K, which is lacunary for a connected piece of F above K is of capacity zero. Since there are at most enumerably infinite number of such connected pieces above K, the set E in K, which is lacunary for some connected piece of F above K is of capacity zero. Let K_n $(n=1, 2, \ldots)$ be discs on the x-plane, whose centers are rational points and whose radii are rational numbers and E_n be the corresponding set in K_n. Then cap. $E_n = 0$ and hence $E = \sum_{n=1}^{\infty} E_n$ is of capacity zero. E is F_σ, i. e. a sum of enumerably infinite number of closed sets. Let (x_0) be a direct transcendental singularity of $y(x)$. Then x_0 is lacunary for a connected piece above a certain K_n, which contains (x_0) as its boundary. Hence x_0 is contained in E_n and so in E. Hence the set of points on the x-plane, which are the projections of direct transcendental singularities is of capacity zero. Hence we have

Theorem V. Let $G(x, y)$ be an integral function with respect to x and y and $y(x)$ be an analytic function defined by $G(x, y) = 0$. Then the set of points on the x-plane, which are the projections of the direct transcendental singularities of $y(x)$ is of capacity zero.