45. *Banach Limits and the Čech Compactification of a Countable Discrete Set.*

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§ 1. Let (m) be the Banach space of all bounded sequences of real numbers $x=\{x_n \mid n=1, 2, \dots\}$ with $\|x\|=\sup_n |x_n|$ as its norm, and let $\bar{\Omega}$ be the Čech compactification (see § 2) of a countable discrete set Ω. The purpose of this paper is to give further investigations into the relations between Banach limits defined on (m) and countably additive measures defined on $\bar{\Omega}$, which were previously discussed by one of the present authors[1]. By means of this compactification $\bar{\Omega}$, we shall first obtain a general form of bounded linear functionals defined on (m) (Theorem 2). This may be considered as a generalization of a result due to L. W. Cohen and N. Dunford[2]; and a result of G. Fichtenholz and L. Kantorovitch[3] concerning the cardinal number of the conjugate space of (m) may be obtained from this easily (Theorem 3).

It will then be shown that a Banach limit corresponds to a countably additive measure $m(\bar{B})$ defined for all Borel subsets $\bar{B}$ of $\bar{\Omega}$ vanishing identically on Ω. Thus the problems of Banach limits are reduced to the problems of measures on the compact (=bicompact) Hausdorff space $\Omega'=\bar{\Omega}-\Omega$. We shall prove that there exists a family of mutually disjoint non-empty open-and-closed subsets of Ω' with the cardinal number $\mathfrak{c}$ (Theorem 6). As an application of this result we shall finally show that, given a sequence $\{f_n(x) \mid n=1, 2, \dots\}$ of bounded linear functionals defined on a closed linear subspace X_0 of a Banach space X, it is not always possible to extend each $f_n(x)$ to a bounded linear functional $F_n(x)$ defined on X in such a way that the new sequence $\{F_n(x) \mid n=1, 2, \dots\}$ is weakly convergent on the whole space X. In fact, if we take $X=(m)$, $X=(c)=$the subspace of (m) consisting of all convergent sequences $x=\{x_n \mid n=1, 2, \dots\}$, and $f_n(x)=x_n-x_{n+1}$, $n=1, 2, \dots$, then this is a required example (see § 7). As was kindly communicated to the autnors by S. Izumi, this fact was already noticed by R. S. Phillips[4]; but our method of proof is entirely different from his and may be observed with some interest.

§ 2. We begin with preliminary remarks. First we note that (m)

1) S. Kakutani, Concrete representation of an abstract (M)-space and the characterization of the space of continuous functions, Annals of Math., **42** (1941).

2) L. W. Cohen and N. Dunford, Transformation in sequence spaces, Duke Math. Journ., **3** (1939), 689–701.

3) G. Fichtenholz and L. Kantorovitch, Sur les opérations dans l'espace des fonctions bornées, Studia Math., **5** (1934), 69–98.

4) R. S. Phillips, On linear transformations, Trans. Amer. Math. Soc., **44** (1939), 516–541.

may be considered as the Banach space $C(\Omega)$ of all bounded real-valued continuous functions $x(\omega)$ defined on a countable discrete set $\Omega=\{\omega_n \mid n=1, 2, \dots\}$ with $\|x\|=\sup_{\omega\in\Omega}|x(\omega)|=\sup_n|x(\omega_n)|$ as its norm. It is well known that (m) and $C(\Omega)$ may be considered as a real normed ring or as an abstract (M)-space[1] (i. e. a Banach lattice in which $\|x \vee y\|=\max(\|x\|, \|y\|)$ for any pair of elements x and y with $x\geqq 0$ and $y\geqq 0$). By a result of E. Čech[2], there exists a compact Hausdorff space $\bar{\Omega}$ containing Ω as a dense subset such that every bounded real-valued continuous function $x(\omega)$ defined on Ω can be uniquely extended to a real-valued continuous function $x(\bar{\omega})$ on $\bar{\Omega}$. This compact Hausdorff space $\bar{\Omega}$ is called the Čech compactification of Ω. The fact stated above shows that, denoting by $C(\bar{\Omega})$ the Banach space of all real-valued continuous functions $x(\bar{\omega})$ defined on $\bar{\Omega}$ with $\|x\|=\sup_{\bar{\omega}\in\bar{\Omega}}|x(\bar{\omega})|$ as its norm, the spaces $C(\Omega)$ and $C(\bar{\Omega})$ are isometric as Banach spaces, ring isomorphic as real normed rings, and lattice isomorphic as (M)-spaces, all these with respect to the same correspondence $x(\omega)\leftrightarrow x(\bar{\omega})$. Further, it is known that we can obtain the Čech compactification $\bar{\Omega}$ of Ω by considering $C(\Omega)$ as a real normed ring and by taking its representation space in the sense of M. H. Stone[3], G. Šilov[4], I. Gelfand and A. Kolmogoroff[5]; or by considering $C(\Omega)$ as an (M)-space and by taking its concrete representation[6].

It is also not difficult to see that $\bar{\Omega}$ may be considered as a representation space (=Boolean space) in the sense of M. H. Stone[7] of the Boolean algebra $\mathfrak{B}(\Omega)$ of all subsets of Ω. From this follows that $\bar{\Omega}$ is totally disconnected and that the Boolean algebra $\mathfrak{B}(\bar{\Omega})$ of all open-and-closed subsets of $\bar{\Omega}$ is isomorphic with $\mathfrak{B}(\Omega)$ (see the proof of Theorem 4). It is also known[8] that $\bar{\Omega}$ has the cardinal number $2^{\mathfrak{c}}$.

§ 3. Let $\bar{\Omega}$ be the Čech compactification of a countable discrete set Ω as in § 2[9], and let $C(\bar{\Omega})$ be the Banach space of all real-valued continuous functions $x(\bar{\omega})$ defined on $\bar{\Omega}$ with $\|x\|=\sup_{\bar{\omega}\in\bar{\Omega}}|x(\bar{\omega})|$ as its norm. We denote by $\bar{\mathfrak{B}}$ the Borel field of all Borel subsets $\bar{B}$ of $\bar{\Omega}$. A countably additive measure $m(\bar{B})$ defined on $\bar{\mathfrak{B}}$ is *regular* if there exists for any $\bar{B}\in\bar{\mathfrak{B}}$ and for any $\varepsilon>0$, an open subset $\bar{O}$ of $\bar{\mathfrak{B}}$ such

1) See S. Kakutani 1) of p. 224.

2) E. Čech, On bicompact spaces, Annals of Math., **38** (1937), 823–844.

3) M. H. Stone, Boolean rings and general topology, Trans. Amer. Math. Soc., **42** (1937), 53–93.

4) G. Šilov, Ideals and subrings of the ring of continuous functions, C. R. URSS, **22** (1939), 7–10.

5) I. Gelfand and A. Kolmogoroff, On rings of continuous functions on topological spaces, C. R. URSS, **22** (1939), 11–15.

6) See S. Kakutani 1) of p. 224.

7) See M. H. Stone 3).

8) B. Pospisil, Remarks on bicompact spaces, Annals of Math., **38** (1937), 845–846.

9) The results of this § is valid for any compact Hausdorff space.

that $\bar{B} \leqq \bar{O}$ and $m(\bar{O}) < m(\bar{B}) + \varepsilon$. Further, a countably additive real-valued (not-necessarily non-negative) set function $m(\bar{B})$ defined on $\bar{\mathfrak{B}}$ is *regular* if it is the difference of two regular countably additive measures defined on $\bar{\mathfrak{B}}$. This condition is clearly equivalent to saying that there exist, for any $\bar{B} \in \bar{\mathfrak{B}}$ and for any $\varepsilon > 0$, a closed subset $\bar{C}$ of $\bar{\Omega}$ and an open subset $\bar{O}$ of $\bar{\Omega}$ such that $\bar{C} \leqq \bar{B} \leqq \bar{O}$ and $|m(\bar{B}) - m(\bar{B}')| < \varepsilon$ for any $\bar{B}' \in \bar{\mathfrak{B}}$ with $\bar{C} \leqq \bar{B}' \leqq \bar{O}$.

Theorem 1[1]. *For any regular countably additive real-valued set function $m(\bar{B})$ defined on $\bar{\mathfrak{B}}$, the integral*

$$f(x) = \int_{\bar{\Omega}} x(\bar{\omega}) m(d\bar{\omega}) \tag{1}$$

is a bounded linear functional defined on $C(\bar{\Omega})$, satisfying

$$\|f\| = \textit{total variation of} \quad m(\bar{B}) \quad \textit{on} \quad \bar{\Omega}. \tag{2}$$

Conversely, to any bounded linear functional $f(x)$ defined on $C(\bar{\Omega})$, there corresponds a uniquely determined regular countably additive real-valued set function $m(\bar{B})$ defined on $\bar{\mathfrak{B}}$ such that (1) *is satisfied for any $x(\bar{\omega}) \in C(\bar{\Omega})$. This correspondence $f(x) \leftrightarrow m(\bar{B})$ is one-to-one and the property of $f(x)$:*

$$x(\bar{\omega}) \geqq 0 \quad on \quad \bar{\Omega} \rightarrow f(x) \geqq 0 \tag{3}$$

corresponds to the fact that

$$m(\bar{B}) \geqq 0 \quad on \quad \bar{\mathfrak{B}} \tag{4}$$

$\Big($*i.e. to the fact that $m(\bar{B})$ is a measure*$\Big)$ *in which case the condition*

$$x(\bar{\omega}) \equiv 1 \quad on \quad \bar{\Omega} \rightarrow f(x) = 1 \tag{5}$$

is equivalent to saying that $m(\bar{\Omega}) = 1$.

§ 4. A bounded linear functional $f(x)$ defined on the Banach space (m) is a *Banach limit* if it satisfies the following conditions[2]:

$$x_n \geqq 0, \quad n = 1, 2, \dots \rightarrow f(x) \geqq 0, \tag{6}$$

$$x_n = 1, \quad n = 1, 2, \dots \rightarrow f(x) = 1, \tag{7}$$

$$x_n = 0, \quad n = k, k+1, k+2, \dots \rightarrow f(x) = 0 \quad (k = 1, 2, \dots). \tag{8}$$

Let us identify $x = \{x_n \mid n = 1, 2, \dots\} \in (m)$ with $x(\omega) = \{x(\omega_n) \mid n = 1, 2, \dots\} \in C(\Omega)$ and also with its extention $x(\bar{\omega}) \in C(\bar{\Omega})$ by means of the relation $x_n = x(\omega_n)$, $n = 1, 2, \dots$

Theorem 2. For any regular countably additive real-valued set function $m(\bar{B})$ defined on $\bar{\mathfrak{B}}$, the formula

$$f(x) = \int_{\bar{\Omega}} x(\bar{\omega}) m(d\bar{\omega}) = \textstyle\sum_{n=1}^{\infty} \mu_n x_n + \displaystyle\int_{\bar{\Omega} - \Omega} x(\bar{\omega}) m(d\bar{\omega}) \tag{9}$$

1) See S. Kakutani 1) of p. 224.

2) Sometimes it is also required that $x_n = y_{n+1}$, $n = 1, 2, \dots$ implies $f(x) = f(y)$. This condition is stronger than (8). We do not discuss this condition in the present paper.

where $\mu_n = m\big((\omega_n)\big)$ $\big((\omega_n)$ *is a set consisting of a single element* $\omega_n\big)$, *is a bounded linear functional defined on* (m) *satisfying*

(10) $\|f\| = \sum_{n=1}^{\infty} |\mu_n| +$ *total variation of* $m(\bar{B})$ *on* $\bar{\Omega} - \Omega$.

Conversely, for any bounded linear functional $f(x)$ *defined on* (m), *there exists a regular countably additive real-valued set function* $m(\bar{B})$ *defined on* $\bar{\mathfrak{B}}$ *such that* (9) *is satisfied for any* $x(\bar{\omega}) \in C(\bar{\Omega})$ $\big($*i. e. for any* $x \in (m)\big)$. *This correspondence* $f(x) \leftrightarrow m(\bar{B})$ *is one-to-one, and the property* (6) *of* $f(x)$ *corresponds to the property* (4) *of* $m(\bar{B})$ $\big($*i. e. to the fact that* $m(\bar{B})$ *is a measure*$\big)$, *in which case the condition* (7) *on* $f(x)$ *is equivalent to saying that* $m(\bar{\Omega}) = 1$.

Further, the condition (8) *corresponds to the fact that* $\mu_n = m\big((\omega_n)\big) = 0$, $n = 1, 2, \dots$. *Thus a Banach limit* $f(x)$ *corresponds to a regular countably additive measure* $m(\bar{B})$ *defined on* $\bar{\mathfrak{B}}$ *vanishing identically on* Ω *such that* $m(\bar{\Omega} - \Omega) = 1$. *Hence the representation formula* (9) *becomes*

$$f(x) = \int_{\bar{\Omega} - \Omega} x(\bar{\omega}) m(d\bar{\omega}). \tag{11}$$

Conversely, the integral (11) *taken with respect to a regular countably additive measure* $m(\bar{B})$ *defined on* $\bar{\mathfrak{B}}$ *satisfying* $m(\bar{\Omega} - \Omega) = 1$, *gives a Banach limit.*

The proof of Theorem 2 is easy, and so omitted.

The formulae (9) and (10) may be considered as a generalization of a result due to L. W. Cohen and N. Dunford[1] giving a general form of bounded linear functionals on (m).

The results obtained in Theorem 2 show that there exists a one-to-one correspondence between Banach limits $f(x)$ and regular countably additive measures $m(B')$ defined for all Borel subsets B' of a compact Hausdorff space $\Omega' = \bar{\Omega} - \Omega$ satisfying $m(\Omega') = 1$. Consequently, in order to discuss general properties of Banach limits, it is natural to investigate topological properties of the compact Hausdorff spach Ω'.

§ 5. From what we have seen in § 2, it is clear that Ω' is totally disconnected and has the cardinal number $2^{\mathfrak{c}}$. Consequently, by considering a measure on Ω' which is concentrated in a single point of Ω', we see that the family of all Banach limits on (m) has a cardinal number not smaller than $2^{\mathfrak{c}}$. On the other hand, since the Banach space (m) has the cardinal number $\mathfrak{c}$, the cardinal number of the conjugate space of (m) cannot exceed $2^{\mathfrak{c}}$. Consequently,

Theorem 3. The conjugate space of (m) (*i.e. the family of all finitely additive measures defined for all subsets of* Ω) *and the family of all Banach limits on* (m) *have both the same cardinal number* $2^{\mathfrak{c}}$.

This result was already obtained by G. Fichtenholz and L. Kanto-

1) L. W. Cohen and N. Dunford 2) of p. 224.

rovitch[1] by a rather complicated method. Our approach is different from theirs.

§ 6. We shall now show that there exists a family of mutually disjoint non-empty open-and-closed subsets of Ω with the cardinal number $\mathfrak{c}$. This fact is rather surprising since in a larger space $\bar{\Omega}$, every family of mutually disjoint non-empty open-and-closed subsets of $\bar{\Omega}$ is countable. An analogous phenomenon was previously observed by E. Szpilrajn[2]. It is, in fact, to be noticed that every compact Hausdorff space is homomorphic with a closed subset of a compact topological group, and that there exists, for any cardinal number $\mathfrak{m}$, a compact Hausdorff space in which there exists a family of mutually disjoint non-empty open-and-closed subsets with the cardinal number $\mathfrak{m}$ (for example, the Čech compactification of a discrete set with the cardinal number $\mathfrak{m}$), while, on the other hand, in any compact topological group every family of mutually disjoint non-empty open sets is countable, this last fact being an immediate consequence of the existence of Haar measure.

In order to prove our proposition, let us denote by $\mathfrak{B}(\Omega')$ the Boolean algebra of all open-and-closed subsets of Ω'. (The Boolean algebras $\mathfrak{B}(\Omega)$ and $\mathfrak{B}(\bar{\Omega})$ were defined at the end of § 2). Let us further denote by $\mathfrak{F}(\Omega)$ the family of all finite subsets of Ω. Then $\mathfrak{F}(\Omega)$ is an ideal in $\mathfrak{B}(\Omega)$ and

Theorem 4. The factor ring $\mathfrak{B}(\Omega)/\mathfrak{F}(\Omega)$ *is isomorphic with* $\mathfrak{B}(\Omega')$.

Proof. Let E be an arbitrary subset of Ω, and let $x_E(\omega)$ be the characteristic function of E defined on Ω. Then $x_E(\omega)$ is continuous on the discrete set Ω, and consequently there exists a unique continuous extension $x_E(\bar{\omega}) \in C(\bar{\Omega})$. Since $x_E(\omega)$ takes only 0 and 1 on Ω, so does $x_E(\bar{\omega})$ on $\bar{\Omega}$. Consequently, $x_E(\bar{\omega})$ is a characteristic function of a certain subset $\bar{E}$ of $\bar{\Omega}$ which is clearly open-and-closed. It is easy to see that $E \leftrightarrow \bar{E}$ gives the isomorphism between $\mathfrak{B}(\Omega)$ and $\mathfrak{B}(\bar{\Omega})$ stated at the end of § 2, and that we have $E=\bar{E} \cap \Omega$. Further, for any $\bar{E} \in \mathfrak{B}(\bar{\Omega})$, let us put $E'=\bar{E} \cap \Omega'$. Then the mapping $E \rightarrow \bar{E} \rightarrow E'$ is clearly a homomorphism of $\mathfrak{B}(\Omega)$ into $\mathfrak{B}(\Omega')$. It is now not difficult to see that this is even an onto-homomorphism and that the family of all $E \in \mathfrak{B}(\Omega)$ which are mapped by this homomorphism into the zero element of $\mathfrak{B}(\Omega')$ is exactly the ideal $\mathfrak{F}(\Omega)$ in question. Our theorem follows from this immediately.

Theorem 5. There exists a family $\mathfrak{A}=\{A_\gamma \mid \gamma \in \Gamma\}$ *of subsets of a countable set* Ω *with the following properties:*

(12) Γ *has the cardinal number* $\mathfrak{c}$,

(13) A_γ *is infinite for any* $\gamma \in \Gamma$,

(14) $A_\beta \cap A_\gamma$ *is finite for any* $\beta \in \Gamma$, $\gamma \in \Gamma$ *with* $\beta \neq \gamma$.

Proof. We may assume that $\Omega=\{\omega_n \mid n=1, 2, \ldots\}$ is expressed by

1) See G. Fichtenholz and L. Kantorovitch 3) of p. 224.

2) E. Szpilrajn, Remarque sur les produits cartésiens d'espaces topologiques, C. R. URSS, **31** (1941), 525–527.

means of the set of all finite diadic systems: $\Omega=\{\omega_{\epsilon_1,\epsilon_2,\ldots,\epsilon_n}\mid\epsilon_1=0$ or $1;\ i=1,\ldots,n,\ n=1,2,\ldots\}$ and that $\Gamma=\{\gamma\}$ is the set of all diadic infinite sequences $\Gamma=\Big\{\gamma=\{\epsilon_1,\epsilon_2,\ldots,\epsilon_n,\ldots\}\mid\epsilon_i=0$ or $1;\ i=1,2,\ldots\Big\}$. Then $\Big\{A_\gamma=\{\omega_{\epsilon_1},\omega_{\epsilon_1,\epsilon_2},\ldots,\omega_{\epsilon_1,\epsilon_2,\ldots,\epsilon_n},\ldots\}\mid\gamma\in\Gamma\Big\}$ gives a required family. It is clear that the conditions (12), (13) and (14) are all satisfied.

From Theorems 4 and 5 follows immediately

Theorem 6. There exists a family $\mathfrak{E}=\{E'_\gamma\mid\gamma\in\Gamma\}$ *of mutually disjoint non-empty open-and-closed subsets of* Ω' *with the cardinal number* $\mathfrak{c}$.

§ 7. Let now $\{f_n(x)\mid n=1,2,\ldots\}$ be a sequence of bounded linear functionals on the Banach space (c) of all convergent sequences $x=\{x_n\mid n=1,2,\ldots\}$ with $\|x\|=\sup_n|x_n|$ defined by $f_n(x)=x_n-x_{n+1}$, $n=1,2,\ldots$. It is clear that this sequence converges weakly to 0 on (c). We shall show that it is impossible to extend each $f_n(x)$ to a bounded linear functional $F_n(x)$ defined on (m) in such a way that the sequence $\{F_n(x)\mid n=1,2,\ldots\}$ is weakly convergent on (m).

Assume that each $f_n(x)$ had an extension $F_n(x)$ defined on (m). By Theorem 2, each $F_n(x)$ is of the form (9), and from the fact that $F_n(x)$ is an extension of $f_n(x)$ we see:

$$F_n(x)=x_n-x_{n+1}+\int_{\Omega'}x(\bar\omega)m_n(d\bar\omega)\,, \tag{15}$$

where $m_n(\bar B)$ is a regular countably additive real-valued set function defined on $\bar B$. Let $\bar m_n(E'_\gamma)$ be the total variation of $m_n(\bar B)$ on E'_γ. Then $\bar m_n(E'_\gamma)>0$ at most for a countable number of $\gamma\in\Gamma$. Since Γ has the cardinal number $\mathfrak{c}$, there exists a $\gamma_0\in\Gamma$ such that $\bar m_n(E'_{\gamma_0})=0$, $n=1,2,\ldots$.

Let now $\bar E_{\gamma_0}$ be an open-and-closed subset of $\bar\Omega$ which satisfies $E'_{\gamma_0}=\bar E_{\gamma_0}\cap\Omega'$, and let us put[1] $E_{\gamma_0}=\bar E_{\gamma_0}\cap\Omega$. Let $x_{\bar E_{\gamma_0}}(\bar\omega)\in C(\bar\Omega)$ be the characteristic function of $\bar E_{\gamma_0}$. Then the integral on the right hand side of (15) vanishes identically for $x(\bar\omega)=x_{\bar E_{\gamma_0}}(\bar\omega)$. Further, since both E_{γ_0} and $\Omega-E_{\gamma_0}$ are clearly infinite sets, there are infinitely many 1 and -1 in the sequence $\{x_n-x_{n+1}\mid n=1,2,\ldots\}$. Consequently, the right hand side of (15) cannot converge for $x(\bar\omega)=x_{\bar E_{\gamma_0}}(\bar\omega)$, showing the impossibility of extension with the required property.

§ 8. Finally, let us add some words concerning the case when Ω is a discrete set with a cardinal number greater than $\aleph_0$. All arguments in this paper are valid in our case, except those which are concerned with the notion of cardinal numbers and which require suitable modifications. We may also discuss the relations between finitely additive measures defined for all subsets of a discrete set Ω and countably additive measures defined for all Borel subsets of the Čech compactification $\bar\Omega$ of Ω. These problems will be discussed on another occasion.

1) $\bar E_{\gamma_0}$ and E_{γ_0} are determined up to a finite set.