96. *Bohr Compactifications of a Locally Compact Abelian Group I.*

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§ 1. *Introduction.* The purpose of this paper[1] is to give a general theory of Bohr compactifications of a locally compact abelian group. We begin with the discussion of general properties of Bohr compactifications (§ 2). The theory of a monothetic or a solenoidal compact group, which recently became more important because of its newly discovered relations[2] with the theory of ergodic measure preserving transformations or flows[3] having a pure point spectrum, will then be discussed as a special case of our theory (§§ 5, 6). We shall give a new proof to a theorem of A. Weil[4] to the effect that a non-discrete monothetic group is compact whenever it is locally compact (Theorem 5). Among other things it will then be shown that a compact abelian group, whose cardinal number does not exceed 2^c, is a solenoidal group if and only if it is connected (Theorem 6). The existence of a non-separable[5] monothetic or a solenoidal group[6] which once seemed to be rather surprising will now turn out to be quite a natural fact after a general method of taking Bohr compactifications of a locally compact abelian group is obtained (§ 3 and Theorems 7,15). The more detailed investigations of this phenomenon were published in a paper[7] of one of the present authors in which the problems concerning cardinal numbers of a compact abelian group are discussed.

Further, the structure of the universal Bohr compactification of an arbitrary locally compact abelian group, and also the structure of the universal monothetic or the universal solenoidal compact group will be determined by means of the method of character groups (§ 4, 5, 6). This result is closely related with the results obtained in another occasion[8] in connection with the problems of the normed ring of a locally compact abelian group. Finally it is interesting to com-

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1) This paper is divided into two parts: I (§§ 1, 2, 3, 4) and II (§§ 5, 6).

2) P. R. Halmos and J. von Neumann, Annals of Math., **43** (1942).

3) flow=one parameter group of measure preserving transformations.

4) A. Weil: L'intégration dans les groupes et leurs applications, Actualités, Paris, 1939.

5) A topological group (or space) is called separable if it satisfies the second countability axiom of Hausdorff.

6) The infinite direct sum $\sum_\alpha \oplus K_\alpha$ of a continuum number of compact abelian groups K_α, each of which is topologically isomorphic with the additive group of real numbers mod. 1 with the usual compact topology, gives an example of a monothetic (or solenoidal) compact group which is not separable.

7) S. Kakutani, On cardinal numbers related with a compact abelian group, Proc., **19** (1943), 366–372.

8) K. Kodaira and S. Kakutani, Normed ring of a locally compact abelian group, Proc. **19** (1943), 360–365.

pare the theory of Bohr compactifications of a locally compact abelian group with the theory of Čech compactifications[9] of a completely regular Hausdorff space. There is a close analogy between the two theories, but an essential difference lies in the fact that the universal Čech compactification of a discrete Hausdorff space is always totally disconnected, while this is not always the case for the universal Bohr compactification of a discrete abelian group.

2. *Bohr compactifications of a locally compact abelian group.*

Theorem 1. Let H and G be two locally compact abelian groups, and denote by H^* and G^* the locally compact character groups of H and G respectively. If there exists a continuous homomorphism $a'=\varphi(a)$ of H onto a topological subgroup H' of G, then there exists a continuous homomorphism $a^{*\prime}=\varphi^*(a^*)$ of G^* onto a topological subgroup $G^{*\prime}$ of H^*, which is determined by the formula:

$$(1) \qquad \big(\varphi(a), a^*\big)=\big(a, \varphi^*(a^*)\big)$$

for any $a\in H$ and for any $a^*\in G^*$. This relation is self-dual, i.e. $\varphi^{**}(a)=\varphi(a)$. Further, the second homomorphism $a^{*\prime}=\varphi^*(a^*)$ is an isomorphism if and only if the image $\varphi(H)=H'$ of H by the first homomorphism $a'=\varphi(a)$ is dense in G, and the image $\varphi^*(G^*)=G^{*\prime}$ of G^* by the second homomorphism $a^{*\prime}=\varphi^*(a^*)$ is dense in H^* if and only if the first homomorphism $a'=\varphi(a)$ is an isomorphism.

The proof of Theorem 1 is omitted. $\varphi^*(a)$ is called the *adjoint* of $\varphi(a)$. A compact abelian group G is a *Bohr compactification* of a locally compact abelian group H if there exists a continuous homomorphism $a'=\varphi(a)$ of H onto a topological subgroup $\varphi(H)=H'$ of G which is dense in G. From Theorem 1 then follows immediately:

Theorem 2. In order that a compact abelian group G be a Bohr compactification of a locally compact abelian group H, it is necessary and sufficient that the discrete character group G^* of G be algebraically isomorphic with an algebraic subgroup of the locally compact character group H^* of H.

§ 3. *Bohr compactifications and almost periodic functions.* Let H be a locally compact abelian group, and let $f^*(a)$ be an arbitrary complex-valued Bohr almost periodic function defined on H. It is well known that there exists a countable family $X^*(f^*)=\{a^*\}$ of continuous characters a^* on H (or, equivalently, a countable subset $X^*(f^*)$ of the locally compact character group H^* of H) with the following property: for any $\varepsilon>0$, there exis a finite system $\{a_1^*,\dots,a_k^*\}\subseteqq X^*(f^*)$ and a finite system $\{\alpha_1,\dots,\alpha_k\}$ of complex numbers such that

$$(2) \qquad \Big|f^*(a)-\textstyle\sum_{\nu=1}^{k}\alpha_\nu \exp\big(2\pi i(a, a_\nu^*)\big)\Big|<\varepsilon$$

for all $a\in H$; conversely, for any $\varepsilon>0$, there exists a finite system $\{a_1,\dots,a_k\}\subseteqq H$ and a finite system $\{\alpha_1,\dots,\alpha_k\}$ of complex numbers such that

9) E. Čech, Annals of Math., 38 (1937), 823–844.

(3) $$\left|\exp\left(2\pi i(a, a^*)\right)-\sum_{p=1}^{k}\alpha_p f^*(a+a_p)\right|<\varepsilon$$

for all $a \in H$.

Theorem 3. Let H be a locally compact abelian group and let $F^*=\{f^*(a)\}$ be an arbitrary family of complex-valued Bohr almost periodic functions $f^*(a)$ defined on H. Let us introduce a uniform structure (H, V_γ, Γ) on H by

(4) $$V_\gamma=\left\{\{a, b\}\,\middle|\, |f_p^*(a+a_p)-f_p^*(b+a_p)|<\varepsilon,\ p=1,\dots,k\right\},$$

(5) $$\Gamma=\left\{\gamma=\{f_1^*,\dots,f_k^*\,;\ a_1,\dots,a_k\,;\ \varepsilon\}\,\middle|\,\{f_1^*,\dots,f_k^*\}\subseqq F^*,\right.$$
$$\left.\{a_1,\dots,a_k\}\subseqq H,\ k=1,2,\dots\,;\ \varepsilon>0\right\},$$

The uniform structure (H, V_γ, Γ) thus introduced on H is uniformly equivalent with the uniform structure (H, W_δ, Δ) defined on H by

(6) $$W_\delta=\left\{\{a, b\}\,\middle|\, |(a, a_p^*)-(b, a_p^*)|<\varepsilon,\ p=1,\dots,k\right\},$$

(7) $$\Delta=\left\{\delta=\{a_1^*,\dots,a_k^*\,;\ \varepsilon\}\,\middle|\,\{a^*,\dots,a_k^*\}\subseqq X^*(F^*),\ k=1,2,\dots\,;\ \varepsilon>0\right\}$$

where we put $X^*(F^*)=\bigcup_{f^*\in F^*}X^*(f^*)$.

The set N of all $a\in H$ which are not separated from the zero element 0 of H in this uniform structure is a closed invariant subgroup of H, and we obtain a natural induced uniform structure $(H/N, V'_\gamma, \Gamma)$ or $(H/N, W'_\delta, \Delta)$ on the factor group $H'=H/N$ of H by N, with respect to which H' is a totally bounded topological group and such that the natural mapping $a\to\varphi(a)$ of H onto $H/N=H'$ is a continuous homomorphism.

The completion of $H'=H/N$ with respect to this uniform structure then yields a compact abelian group G which is a Bohr compactification of H; conversely, every Bohr compactification of H can be obtained in this way. Further, the discrete character group G^* of G is algebraically isomorphic with the algebraic subgroup X^* of H^* which is generated by $X^*(F^*)=\bigcup_{f^*\in F^*}X^*(f^*)$, and so G is separable if and only if X^* or $X^*(F^*)$ is a countable set.

Finally, for any complex valued continuous (and hence Bohr almost periodic) function $x^*(a)$ defined on G, $f^*(a)=x^*\left(\varphi(a)\right)$ is a complex valued Bohr almost periodic function defined on H which can be uniformly approximated on H by a finite linear combination of exponential character $\exp\left(2\pi i(a, a^*)\right)$, $a^*\in X^*(F^*)$; conversely, every such Bohr almost periodic function $f^*(a)$ on H (and hence every Bohr almost periodic function $f^*(a)\in F^*$) can be obtained in this way, i.e. for any such Bohr almost periodic function $f^*(a)$ defined on H, there exists a complex-valued continuous function $x^*(a)$ defined on G such that $f^*(a)=x^*\left(\varphi(a)\right)$ for all $a\in H$.

Remark 1. The uniform structure (H, V_γ, Γ) defined above is uniformly equivalent with the uniform structure (H, d_{f*}, F^*) which is defined on H by means of a system $\{d_{f*} \mid f^* \in F^*\}$ of quasi-metrics:

$$d_{f*}(a, b) = \sup_{h \in H} |f^*(a+h) - f^*(b+h)|. \tag{8}$$

In this form, our result may be considered as a generalization of the results of A. Weil.[10)]

Remark 2. Let X^* be an arbitrary algebraic subgroup of H^*, and let $F^*(X^*) = \{f^*(a)\}$ be the family of all complex-valued Bohr almost periodic functions $f^*(a)$ such that $X^*(f^*) \subseteqq X^*$ (this means that $f^*(a)$ can be uniformly approximated by a finite linear combination of the exponential characters $\exp\big(2\pi i(a, a^*)\big)$, $a^* \in X^*$). Then $F^*(X^*)$ is a normed ring with respect to the ordinary operations of addition, multiplication and scalar multiplication and with respect to the usual norm:

$$\|f^*\| = \sup_{a \in G} \|f^*(a)|. \tag{9}$$

Further, $F^*(X^*)$ is closed under the operation of taking a convolution:

$$f^* \times g^*(a) = M_b\big(f^*(a-b)g^*(b)\big), \tag{10}$$

where $M_b\big(f(b)\big)$ denotes the mean in the sense of J. von Neumann of an almost periodic function $f(b)$. It is then easy to see that $F^*(X^*)$ is isometrically isomorphic with the normed ring of all complex-valued continuous functions defined on a Bohr compactification G of H which is topologically isomorphic with the compact character group $X^{*(d)*}$ of a discrete abelian group $X^{*(d)}$ algebraically isomorphic with X^*, and which can be obtained from H by means of a method described in Theorem 3. We shall call G the *X^*-compactification* of H.

In this form, our results may be considered as a generalization of the results of E. R. van Kampen[11)] concerning moduls of almost periodic functions. Thus there exists a one-to-one correspondence among subgroups of the character group H^* of H, Bohr compactifications of H, and moduls of Bohr almost periodic functions defined on H.

Remark 3. Let X_1^* and X_2^* be two algebraic subgroups of H, and let G_1 and G_2 be the corresponding Bohr compactifications of H. Let us further denote by H_1' and H_2' the dense subgroups of G_1 and G_2 which are the continuous homomorphic images of H such that G_1 and G_2 are obtained from H_1' and H_2' by completion. Then it is easy to see that X_1^* is a subgroup of X_2^* if and only if there exists a continuous homomorphism of G_2 onto G_1 by which every element on H_2' is mapped onto the corresponding element of H_1'.

4. *The universal Bohr compactification of a locally compact abelian group.* Let us now take as X^* the character group H^* of H itself (or, what amounts to the same thing, take as F^* the family of all

10) A. Weil: C. R. Paris, **196** (1936).

11) E. R. van Kampen, Annals of Math., **37** (1936).

complex-valued Bohr almost periodic functions $f^*(a)$ defined on H). Then we see

Theorem 4. For any locally compact abelian group H there exists a Bohr compactification $G=\bar{H}$ of H with the following property: There exists a continuous isomorphism $a'=\varphi(a)$ of H onto a dense subgroup H' of G such that for any complex-valued Bohr almost periodic function $f^*(a)$ defined on H there exists a complex-valued continuous function $x^*(a)$ dedefined on G such that $f^*(a)=x^*(\varphi(a))$ for all $a \in H$. Further, for any Bohr compactification G_1 of H, there exists a continuous homomorphism of G onto G_1 which maps every element of H' onto the corresponding element of H', where we denote by H' a dense subgroup of G_1 which is a continuous homomorphic image of H such that the compactification G_1 is obtained from H'_1 by completion. Finally, $G=\bar{H}$ is topologically isomorphic with the compact character group $H^{*(d)*}$ of a discrete abelian group $H^{*(d)}$ which itself is algebraically isomorphic with the locally compact character group H^* of H.

This Bohr compacification $G=\bar{H}$ of H is called the *universal Bohr compactification* of H. The universal Bohr compactification $\bar{H}$ of a locally compact abelian group H corresponds to the universal Čech compactification Ω of a completely regular Hausdorff space Ω in the theory of topological spaces. The universal Čech compactification Ω of a discrete space Ω is always totally disconnected while it is possible to find a discrete abelian group H whose universal Bohr compactification is connected. In fact it suffices to take as H an abelian group H such that the compact character group H^* of H has no element of finite order. (For example, take as H the additive group of all real numbers with discrete topology).