91. *Conformal and Concircular Geometries in Einstein Spaces.*

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S. Sasaki[1] has recently studied the spaces with normal conformal connexions whose groups of holonomy fix a point or a hypersphere, and derived the fundamental theorem: If the group of holonomy of a space C_n with a normal conformal connexion is a subgroup of the Möbius group which fixes a point (or a hypersphere), the C_n is a space with a normal conformal connexion corresponding to the class of Riemann spaces conformal to each other including an Einstein space with a vanishing (or non-vanishing) scalar curvature. The converse is also true.

But, it seems to me that, the group of holonomy of C_n fixing a point or a hypersphere, the whole space C_n is not necessarily conformal to an Einstein space, but it may admit of an exceptional point or hypersurface. The first purpose of this Note is to study such exceptional cases.

S. Sasaki[1] has also studied the spaces with normal conformal connexions whose groups of holonomy fix two points or hyperspheres. These spaces are closely related to the Einstein spaces which admit a concircular transformation[2]. The second purpose of this Note is to consider the relations between the conformal and the concircular geometries in these spaces.

§ 1. *Spaces whose groups of holonomy fix a point or a hypersphere.*

Let us consider a space C_n with a normal conformal connexion and take the Veblen repere $[A_0, A_\lambda, A_\infty]$[3] in each tangent space, then, the normal conformal connexion may be expressed by the following formulae:

$$\text{(1.1)} \qquad \begin{cases} dA_0 = \qquad\qquad\qquad dx^\lambda A_\lambda\,, \\ dA_\mu = \Pi^0_{\mu\nu}dx^\nu A_0 + \Pi^\lambda_{\mu\nu}dx^\nu A_\lambda + \Pi^\infty_{\mu\nu}dx^\nu A_\infty\,, \\ dA_\infty = \qquad\qquad \Pi^\lambda_{\infty\nu}dx^\nu A_\lambda\,, \end{cases}$$

where

1) S. Sasaki: On the spaces with normal conformal connexions whose groups of holonomy fix a point or a hypersphere, I. Japanese Journal of Mathematics, vol. **18** (1943), pp. 615–622, II. ibidem, pp. 623–633. These papers will be cited as S. I. and S. II. respectively.

2) K. Yano: Concircular geometry I. Concircular transformations, Proc. **16** (1940), 195–200; II. Integrability conditions of $\rho_{\mu\nu}=\phi g_{\mu\nu}$, ibidem, pp. 354–360; III. Theory of curves, ibidem, pp. 442–448; IV. Theory of subspaces, ibidem, pp. 505–511; V. Einstein spaces, ibidem, **18** (1942), pp. 446–451.

3) K. Yano: Sur la théorie des espaces à connexion conforme, Journal of the Faculty of Science, Imperial University of Tokyo, vol. 4, part 1 (1939), pp. 1–59.

The greek indices run from 1 to n and the latin ones from $\dot{1}$ to $\dot{n}-\dot{1}$.

$$
(1.2)\qquad \begin{cases} \Pi^0_{\mu\nu} = -\dfrac{R_{\mu\nu}}{n-2} + \dfrac{Rg_{\mu\nu}}{2(n-1)(n-2)}, \\ \Pi^\lambda_{\mu\nu} = \{{}^{\lambda}_{\mu\nu}\} = \dfrac{1}{2} g^{\lambda\alpha}\left(\dfrac{\partial g_{\alpha\mu}}{\partial x^\nu} + \dfrac{\partial g_{\alpha\nu}}{\partial x^\mu} - \dfrac{\partial g_{\mu\nu}}{\partial x^\alpha}\right), \\ \Pi^\infty_{\mu\nu} = g_{\mu\nu}, \qquad \Pi^\lambda_{\infty\nu} = g^{\lambda\mu}\Pi^0_{\mu\nu}, \end{cases}
$$

$R_{\mu\nu}$ and R being respectively the Ricci tensor and scalar curvature formed with the fundamental tensor $g_{\mu\nu} = A_\mu A_\nu$. We suppose that the fundamental quadratic form $g_{\mu\nu}dx^\mu dx^\nu$ is positive definite.

The normal conformal connexion of C_n being thus defined, we suppose that the group of holonomy of C_n fixes a point or a hypersphere,

$$
(1.3)\qquad \phi = \phi^0 A_0 + \phi^\lambda A_\lambda + \phi^\infty A_\infty,
$$

then we must have $d\phi = \phi\tau_\nu dx^\nu$, from which we obtain

$$
(1.4)\qquad \begin{cases} (a) \quad \phi^0_{,\nu} + \Pi^0_{\mu\nu}\phi^\mu = \phi^0\tau_\nu, \\ (b) \quad \phi^0\delta^\lambda_\nu + \phi^\lambda_{;\nu} + \Pi^\lambda_{\infty\nu}\phi^\infty = \phi^\lambda\tau_\nu, \\ (c) \quad \phi^\infty_{,\nu} + \phi_\nu = \phi^\infty\tau_\nu, \qquad (\phi_\lambda = g_{\lambda\mu}\phi^\mu) \end{cases}
$$

where the comma and the semi-colon denote respectively the ordinary and covariant derivatives with respect to the Christoffel symbols $\Pi^\lambda_{\mu\nu} = \{{}^{\lambda}_{\mu\nu}\}$.

Here, we must distinguish two types of point or hypersphere as whether there exist or not in the space the points for which $\phi^\infty = 0$.

Type (A), (i). There exists a point for which $\phi^\infty = 0$, ϕ being a point-sphere.

The ϕ being a point-sphere, we have $g_{\mu\nu}\phi^\mu\phi^\nu - 2\phi^0\phi^\infty = 0$, consequently, we must have $\phi^\lambda = 0$ at the point for which $\phi^\infty = 0$. Thus, there exists generally only one such point in C_n and there the point-sphere ϕ reduces to $\phi^0 A_0$, that is to say, the point-sphere ϕ coincides with the point of contact of C_n with the tangent Möbius space. Thus we have the

Theorem I.I. If the group of holonomy of C_n fixes a point-sphere of the type (A), there exists, in C_n, a point at which the point-sphere and the point of contact of C_n with the tangent Möbius space coincide. If we map the tangent Möbius space at that point along any closed curve starting from and arriving at the point, the image of the point coincides with the original point. (Cf. S. II. Theorem 8.)

Type (A), (ii). There exist the points for which $\phi^\infty = 0$, ϕ being a hypersphere.

Such points constitute in general a hypersurface which we denote by the parametric representations $x^\lambda = x^\lambda(x^i)$. Then, from the identical relation $\phi^\infty\big(x^\lambda(x^i)\big) = 0$, we have by differentiation

$$
(1.5)\qquad \phi^\infty_{,\lambda} B^{\cdot\lambda}_{i} = 0 \quad \text{and} \quad \phi^\infty_{;\mu;\nu} B^{\cdot\mu}_{j} B^{\cdot\nu}_{k} + \phi^\infty_{,\lambda} H^{\cdot\cdot\lambda}_{jk} = 0,
$$

where $\quad B^{\cdot\lambda}_{i} = \dfrac{\partial x^\lambda}{\partial x^i} \qquad H^{\cdot\cdot\lambda}_{jk} = \dfrac{\partial^2 x^\lambda}{\partial x^j \partial x^k} + B^{\cdot\mu}_{j} B^{\cdot\nu}_{k}\{{}^{\lambda}_{\mu\nu}\} - B^{\cdot\lambda}_{i}\{{}^{i}_{jk}\}$

and $\{{}^{\,i}_{jk}\}$ are Christoffel symbols formed with $g_{jk}=g_{\mu\nu}B_j^{\cdot\mu}B_k^{\cdot\nu}$.

Differentiating (1.4) (c) covariantly and substituting, in the resulting equations, (1.4) (b) written in the covariant form $\phi^0 g_{\mu\nu}+\phi_{\mu;\nu}+\Pi^0_{\mu\nu}\phi^\infty=\phi_\mu\tau_\nu$, we have

$$\phi^\infty_{;\mu;\nu}+(\phi_\mu\tau_\nu-\phi^0 g_{\mu\nu}-\Pi^0_{\mu\nu}\phi^\infty)=\phi^\infty_{,\nu}\tau_\mu+\phi^\infty\tau_{\mu;\nu}\,. \tag{1.6}$$

Contracting $B_j^{\cdot\mu}B_k^{\cdot\nu}$ to (1.4) (c) and (1.6), and remembering the relations $\phi^\infty=0$ and (1.5), we have

$$\phi_\lambda B_j^{\cdot\lambda}=0 \quad\text{and}\quad \phi^\infty_{,\lambda}H_{jk}^{\cdot\cdot\lambda}+\phi^0 g_{jk}=0\,, \tag{1.7}$$

from which and (1.5), we can conclude that the hypersphere ϕ is tangent to the hypersurface $\phi^\infty=0$ and that $H_{jk}^{\cdot\cdot\lambda}=\frac{1}{m}g^{ab}H_{ab}^{\cdot\cdot\lambda}g_{jk}$, say, the hypersurface $\phi^\infty=0$ is totally umbilical. Thus we have the

Theorem I.II. If the group of holonomy of C_n fixes a hypersphere ϕ of the type (A), there exists in C_n at least one totally umbilical hypersurface, and the hypersphere ϕ is always tangent to the hypersurface. (Cf. S. II. Theorem 7.)

The group of holonomy of C_n fixing always a point or a hypersphere, we shall now consider the region of C_n where $\phi^\infty\neq 0$. For such a region, putting $\rho^0=\phi^0/\phi^\infty$ and $\rho^\lambda=\phi^\lambda/\phi^\infty$, the equations (1.4) may be reduced to

$$\left\{\begin{array}{ll}(a) & \rho^0_{,\nu}-\rho^0\rho_\nu+\Pi^0_{\mu\nu}\rho^\mu=0\,,\\ (b) & \rho^0 g_{\mu\nu}+\rho_{\mu;\nu}-\rho_\mu\rho_\nu+\Pi^0_{\mu\nu}=0\,. \qquad (\rho_\lambda=g_{\lambda\mu}\rho^\mu)\end{array}\right. \tag{1.8}$$

The tensors $g_{\mu\nu}$ and $\Pi^0_{\mu\nu}$ being both symmetric with respect to two lower indices μ and ν, the equations (1.8) (b) show that ρ_ν is a gradient vector, say, there exists a function ρ such that $\rho_\nu=\partial\log\rho/\partial x^\nu$. Effecting a conformal transformation $\bar g_{\mu\nu}=\rho^2 g_{\mu\nu}$, we have

$$\bar\Pi^0_{\mu\nu}=\Pi^0_{\mu\nu}+\rho_{\mu;\nu}-\rho_\mu\rho_\nu+\frac{1}{2}g^{\alpha\beta}\rho_\alpha\rho_\beta g_{\mu\nu}\,,$$

and the equations (1.8) (b) written in the form

$$\Pi^0_{\mu\nu}+\rho_{\mu;\nu}-\rho_\mu\rho_\nu+\frac{1}{2}g^{\alpha\beta}\rho_\alpha\rho_\beta g_{\mu\nu}=\Big(\frac{1}{2}g^{\alpha\beta}\rho_\alpha\rho_\beta-\rho^0\Big)g_{\mu\nu}$$

show that

$$\bar\Pi^0_{\mu\nu}=\frac{1}{\rho^2}\Big(\frac{1}{2}g^{\alpha\beta}\rho_\alpha\rho_\beta-\rho^0\Big)\bar g_{\mu\nu}\,. \tag{1.9}$$

The tensor $\bar\Pi^0_{\mu\nu}$ being defined by

$$\bar\Pi^0_{\mu\nu}=-\frac{\bar R_{\mu\nu}}{n-2}+\frac{\bar R\bar g_{\mu\nu}}{2(n-1)(n-2)}\,,$$

the equations (1.9) show that the Riemannian space with fundamental quadratic form $\bar g_{\mu\nu}dx^\mu dx^\nu$ is an Einstein space, say, for which we have $\bar R_{\mu\nu}=\frac{1}{n}\bar R\bar g_{\mu\nu}$, thus we have $\bar\Pi^0_{\mu\nu}=\bar c\bar g_{\mu\nu}$, where $\bar R$ and $\bar c=-\frac{1}{2n(n-1)}\bar R$ are constants. From

(1.10) $$\bar{c}=\frac{1}{\rho^2}\left(\frac{1}{2}g^{\alpha\beta}\rho_\alpha\rho_\beta-\rho^0\right),$$

we have by covariant differentiation

$$-2\rho_\nu\left(\frac{1}{2}g^{\alpha\beta}\rho_\alpha\rho_\beta-\rho^0\right)+(g^{\alpha\beta}\rho_{\alpha;\nu}\rho_\beta-\rho^0_{.\nu})=0\,.$$

Substituting (1.8) (*b*) in these equations, we obtain

$$\rho^0_{.\nu}-\rho^0\rho_\nu+\Pi^0_{\mu\nu}\rho^\mu=0\,,$$

that is to say, the equations (1.8) (*a*) are identically satisfied.

The ϕ being a point or a hypersphere according as $\frac{1}{2}g^{\alpha\beta}\rho_\alpha\rho_\beta-\rho^0=0$ or not, the equation (1.10) shows that the ϕ is a point or a hypersphere according as $\bar{c}=-\frac{1}{2n(n-1)}\bar{R}$ is zero or not. Consequently we have the

Theorem I.III. If the gronp of holonomy of C_n fixes a point of the type (A), C_n is conformal, with the exception of a point, to an Einstein space with vanishing scalar curvature. (Cf. S. I. Fundamental theorem.)

Theorem I.IV. If the group of holonomy of C_n fixes a hypersphere of the type (A), C_n is conformal, with the exception of a totally umbilical hypersurface, to an Einstein space with non-vanishing scalar curvature. (Cf. S. I. Fundamental theorem.)

Type (B). There exists no point for which $\phi^\infty=0$, ϕ being a point-sphere or a hypersphere.

In this case, the above reasoning gives the

Theorem I.V. If the group of holonomy of C_n fixes a point-sphere of the type (B), the whole C_n is conformal to an Einstein space with vanishing scalar curvature. (Cf. S. I. Fundamental theorem.)

Theorem I.VI. If the group of holonomy of C_n fixes a hypersphere of the type (B), the whole C_n is conformal to an Einstein space with non-vanishing scalar curvature. (Cf. S. I. Fundamental theorem.)

Conversely we have the

Theorem I.VII. The group of holonomy of a C_n conformal to an Einstein space with vanishing scalar curvature fixes always a point-sphere of the type (B). (Cf. S. I. Fundamental theorem.)

Theorem I.VIII. The group of holonomy of a C_n conformal to an Einstein space with non-vanishing scalar curvature fixes always a hypersphere of the type (B). (Cf. S. I. Fundamental theorem.)

If we calculate the coefficients of the normal conformal connexion (1.1) in the case of an Einstein space for which $R_{\mu\nu}=\frac{1}{n}Rg_{\mu\nu}$, we have

(1.11) $$\Pi^0_{\mu\nu}=cg_{\mu\nu} \quad\text{and}\quad \Pi^\lambda_{\infty\nu}=c\delta^\lambda_\nu\,,$$

where $$c=-\frac{1}{2n(n-1)}R\,,$$

hence, we can see that the group of holonomy of this normal conformal connexion fixes a point-sphere A_∞ or a hypersphere $-cA_0+A_\infty$ according as the scalar curvature R and consequently c is zero or not.

Now, we shall consider the integrability condition of the equations (1.8). We have, after some straightforward calculation,

$$C^0_{\mu\nu\omega}-\rho_\lambda C^{\lambda}_{\cdot\mu\nu\omega}=0\,, \tag{1.12}$$

where $C^0_{\mu\nu\omega}$ and $C^{\lambda}_{\cdot\mu\nu\omega}$ are respectively the conformal tensors of J. M. Thomas and H. Weyl. Hence, we have the

Theorem I.IX. If the group of holonomy of a C_3 fixes a point or a hypersphere, the C_3 is necessarily conformally flat. (Cf. S. I. Theorem 2.)

§ 2. *Conformal circles.*

In a previous paper[1], we have defined the conformal circles in a conformally connected manifold by the equation $\frac{d^2}{dt^2}\rho A_0=0$, where t is a projective parameter. The differential equations of the conformal circles are

$$\frac{\delta^3 x^\lambda}{\delta s^3}+\frac{dx^\lambda}{ds}g_{\mu\nu}\frac{\delta^2 x^\mu}{\delta s^2}\frac{\delta^2 x^\nu}{\delta s^2}-\frac{dx^\lambda}{ds}\Pi^0_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}+\Pi^{\lambda}_{\infty\nu}\frac{dx^\nu}{ds}=0 \tag{2.1}$$

where s is the arc length. Moreover, we have defined the geodesic circles and conformal geodesic circles[2] by the equations

$$\frac{\delta^3 x^\lambda}{\delta s^3}+\frac{dx^\lambda}{ds}g_{\mu\nu}\frac{\delta^2 x^\mu}{\delta s^2}\frac{\delta^2 x^\nu}{\delta s^2}=0\,, \tag{2.2}$$

and

$$\frac{\delta^3 x^\lambda}{\delta s^3}+\frac{dx^\lambda}{ds}g_{\mu\nu}\frac{\delta^2 x^\mu}{\delta s^2}\frac{\delta^2 x^\nu}{\delta s^2}+\frac{dx^\lambda}{ds}\rho_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}-\rho^{\lambda}_{\cdot\nu}\frac{dx^\nu}{ds}=0 \tag{2.3}$$

respectively, where

$$\rho_{\mu\nu}=\rho_{\mu;\nu}-\rho_\mu\rho_\nu+\frac{1}{2}g^{\alpha\beta}\rho_\alpha\rho_\beta \quad\text{and}\quad \rho^{\lambda}_{\cdot\nu}=g^{\lambda\mu}\rho_{\mu\nu}\,.$$

From these equations, we have the

Theorem II.I. If all conformal circles are geodesic circles, then $\Pi^{\lambda}_{\infty\nu}=c\delta^{\lambda}_{\nu}$, that is to say, the space is an Einstein space[3].

Theorem II.II. If all the conformal circles belong to a system of conformal geodesic circles, then $\Pi^{\lambda}_{\infty\nu}+\rho^{\lambda}_{\cdot\nu}=c\delta^{\lambda}_{\nu}$, that is to say, the space is conformal to an Einstein space.

Now, suppose that the group of holonomy of C_n fixes two points or hyperspheres $P=p^0A_0+p^\lambda A_\lambda+A_\infty$ and $Q=q^0A_0+q^\lambda A_\lambda+A_\infty$ of the type (*B*), and consider a curve $A_0(r)$ whose direction is orthogonal to the hypersphere belonging to the pencil of spheres determined by P and Q and passing through the point on the curve. Then we have the equations of the form

$$\frac{dA_0}{dr}=P-Q\,,\qquad \frac{d^2A_0}{dr^2}=aP+bQ\,,\qquad \frac{d^3A_0}{dr^3}=cP+dQ\,,$$

1) K. Yano: Sur les circonférences généralisées dans les espaces à connexion conforme. Proc. **14** (1938), 329–332.

2) K. Yano: Concircular geometry I. loc. cit.

3) T. Suguri: On the circles in a Riemann space and in a conformally connected manifold. (japanese), Tokyo-Buturigakko-Zassi, **51** (1942), pp. 148-159.

and hence
$$\frac{d^3A_0}{dr^3}+\alpha\frac{d^2A_0}{dr^2}+\beta\frac{dA_0}{dr}=0\,,$$
$\frac{d}{dr}P$ and $\frac{d}{dr}Q$ being proportional to P and Q respectively. This equation may be reduced, by a suitable choice of t and ρ, to the form $\frac{d^3}{dt^3}\rho A_0=0$. Hence we have

Theorem II.III. If the group of holonomy of C_n fixes two points or hyperspheres, the curve whose direction is orthogonal to the hyperspheres belonging to the pencil determined by these points or hyperspheres and passing through the point on the curve is a conformal circle.

§ 3. *Totally umbilical hypersurfaces.*

If we define the repere $[A_{\dot{0}}, A_i, A_{\dot{n}}, A_{\dot{\infty}}]$[1] on a hypersurface C_{n-1}: $x^\lambda=x^\lambda(x^i)$ by

$$\text{(3.1)}\quad \begin{cases} A_{\dot{0}}= A_0\,, \\ A_i= B_i^{\cdot\lambda}A_\lambda\,, \\ A_{\dot{n}}= \dfrac{1}{n-1}H^{a}_{\cdot a\dot{n}}A_0 + B_{\dot{n}}^{\cdot\lambda}A_\lambda\,, \\ A_{\dot{\infty}}=\dfrac{1}{2(n-1)^2}H^{a}_{\cdot a\dot{n}}H^{b}_{\cdot b\dot{n}}A_0+\dfrac{1}{n-1}H^{a\cdot\lambda}_{\cdot a}A_\lambda+A_\infty\,, \end{cases}$$

where $B_{\dot{n}}^{\cdot\lambda}$ is defined by $g_{\mu\nu}B_j^{\cdot\mu}B_{\dot{n}}^{\cdot\nu}=0$ and $g_{\mu\nu}B_{\dot{n}}^{\cdot\mu}B_{\dot{n}}^{\cdot\nu}=1$, and $H_{jk}^{\cdot\cdot\lambda}=H_{jk\dot{n}}B_{\dot{n}}^{\cdot\lambda}$, $A_{\dot{n}}$ being the so-called central sphere, the normal conformal connexion is expressed by the formulae

$$\text{(3.2)}\quad \begin{cases} dA_{\dot{0}}= dx^iA_i\,, \\ dA_j=\Pi^{\dot{0}}_{jk}dx^kA_{\dot{0}}+\Pi^{i}_{jk}dx^kA_i+\Pi_{jk\dot{n}}dx^kA_{\dot{n}}+\Pi^{\dot{\infty}}_{jk}dx^kA_{\dot{\infty}}\,, \\ dA_{\dot{n}}=\Pi^{\dot{0}}_{\dot{n}k}dx^kA_{\dot{0}}+\Pi^{i}_{\dot{n}k}dx^kA_i\,, \\ dA_{\dot{\infty}}= \Pi^{i}_{\dot{\infty}k}dx^kA_i+\Pi_{\dot{\infty}\dot{n}k}dx^kA_{\dot{n}}\,, \end{cases}$$

where

$$\text{(3.3)}\quad \begin{cases} \Pi^{\dot{0}}_{jk}=\Pi^{0}_{\mu\nu}B_j^{\cdot\mu}B_k^{\cdot\nu}-\dfrac{1}{n-1}H^{a}_{\cdot a\dot{n}}H_{jk\dot{n}}+\dfrac{1}{2(n-1)^2}H^{a}_{\cdot a\dot{n}}H^{b}_{\cdot b\dot{n}}g_{jk}\,, \\ \Pi^{i}_{jk}=\left\{{i \atop jk}\right\}\,,\quad \Pi_{jk\dot{n}}=M_{jk\dot{n}}=H_{jk\dot{n}}-\dfrac{1}{n-1}g^{ab}H_{ab\dot{n}}g_{jk}\,,\quad \Pi^{\dot{\infty}}_{jk}=g_{jk}\,, \\ \Pi^{\dot{0}}_{\dot{n}k}=\dfrac{1}{n-2}M^{a}_{\cdot k\dot{n};a}\,,\qquad \Pi^{i}_{\dot{n}k}=-M^{i}_{\cdot k\dot{n}}=-g^{ij}M_{jk\dot{n}}\,, \\ \Pi^{i}_{\dot{\infty}k}=g^{ij}\Pi^{\dot{0}}_{jk}\,,\qquad \Pi_{\dot{\infty}\dot{n}k}=\Pi^{\dot{0}}_{\dot{n}k}\,. \end{cases}$$

Hence, for a totally umbilical hypersurface ($M_{jk\dot{n}}=0$), we have

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1) K. Yano and Y. Muto: Sur la théorie des espaces à connexion conforme normale et la géométrie conforme des espaces de Riemann, Journal of the Faculty of Science, Imperial University of Tokyo. vol. 4 part 3 (1941), pp. 117–169.

$$
(3.4)\quad
\begin{cases}
dA_{\dot 0} = dx^i A_i\,, \\
dA_j = \Pi^{\dot 0}_{jk} dx^k A_{\dot 0} + \Pi^{\dot i}_{jk} dx^k A_i + \Pi^{\dot\infty}_{jk} dx^k A_{\dot\infty}\,, \\
dA_{\dot n} = 0 \\
dA_{\dot\infty} = \Pi^{\dot i}_{\infty k} dx^k A_i\,.
\end{cases}
$$

The conformal connexion induced on C_n being defined by just the same equations as above, we have the

Theorem III.I. Along a totally umbilical hypersurface, the normal conformal connexion of C_n and the conformal connexion induced on C_{n-1} coincide.

It must be however remarked that the conformal connexion induced on C_{n-1} and the intrinsic normal conformal connexion of C_n do not necessarily coincide even if the C_n is totally umbilical. The necessary and sufficient condition that the induced and intrinsic connexions of C_{n-1} coincide is that the C_n is totally umbilical and $B^{\cdot\lambda}_{\underset{n}{\cdot}} B^{\cdot\mu}_{j} B^{\cdot\nu}_{k} B^{\cdot\omega}_{\underset{n}{\cdot}} C_{\lambda\mu\nu\omega} = 0$.

From the third equation of (3.4), we have the

Theorem III.II. Along a totally umbilical hypersurface, the central sphere rests fix by the normal conformal connexion of the enveloping space C_n.

Conversely, if a hypersurface $\alpha A_{\dot 0} + \beta A_{\dot n}$ tangent to a hypersurface rests fix along the hypersurface by the normal conformal connexion of the enveloping space C_n, we have

$$
d(\alpha A_{\dot 0} + \beta A_{\dot n}) = (\alpha A_{\dot 0} + \beta A_{\dot n})\tau_k dx^k\,,
$$

from which, we obtain, by the use of (3.2), $\Pi^{\dot i}_{nk} = 0$, hence, we have the

Theorem III.III. If a hypersphere tangent to a hypersurface rests fix along the hypersurface by the normal conformal connexion of the enveloping space C_n, the hypersurface is totally umbilical.

This theorem gives an another proof of Theorem I.II. For if the group of holonomy of C_n fixes a hypersphere $\phi = \phi^0 A_0 + \phi^\lambda A_\lambda + \phi^\infty A_\infty$, the equations (1.4) (c) show that the hypersphere ϕ is always tangent to the hypersurface $\phi^\infty = 0$, and by the above theorem, we can conclude that the hypersurface $\phi^\infty = 0$ is totally umbilical.

§ 4. *The line-elements of the Riemannian spaces which admit a family of ∞^1 totally umbilical hypersurfaces.*

If a Riemannian space admits a family of ∞^1 totally umbilical hypersurfaces, the line-element of the space may be, by a suitable choice of the coordinate system, put in the form

$$
(4.1)\qquad ds^2 = f(x^\lambda)^* g_{jk}(x^i) dx^j dx^k + g_{nn}(x^\lambda) dx^n dx^n\,,
$$

where the totally umbilical hypersurfaces are x^n = consts. and the x^n-curves are orthogonal trajectories of the family of totally umbilical hypersurfaces.

By a suitable conformal transformation of the fundamental tensor, (4.1) may be reduced to

$$(4.2)\qquad ds^2=f(x^\lambda)^*g_{jk}(x^i)dx^jdx^k+dx^ndx^n\,,$$

where the hypersurfaces $x^n=$consts. being always totally umbilical and x^n-curves, which are their orthogonal trajectories, are geodesics.

If the x^n-curves in (4.1) are conformal circles, the x^n-curves in (4.2) are also conformal circles, but they are also geodesics, hence, from (2.1), they must be Ricci curves, and (4.2) must be of the form[1)]

$$(4.3)\qquad ds^2=\sigma(x^n)^*g_{jk}(x^i)dx^jdx^k+dx^ndx^n\,,$$

which is the line-element of the space which admits concircular transformations. Thus we obtain the

Theorem IV.I. The Riemannian spaces which contain a family of ∞^1 totally umbilical hypersurfaces whose orthogonal trajectories are conformal circles are conformal to the Riemannian spaces which admit concircular transformations.

§ 5. *Spaces whose groups of holonomy fix two points or hyperspheres.*

Suppose that the group of holonomy of C_n fixes two points or hyperspheres of the type (B): $P=p^0A_0+p^\lambda A_\lambda+A_\infty$ and $Q=q^0A_0+q^\lambda A_\lambda+A_\infty$, then from § 1, we know that $p_\lambda=g_{\lambda\mu}p^\mu$ and $q_\lambda=g_{\lambda\mu}q^\mu$ are both gradient vectors such as $p_\lambda=\partial\log p/\partial x^\lambda$ and $q_\lambda=\partial\log q/\partial x^\lambda$. Consequently, when we displace on the hypersurfaces defined by $p/q=$consts., the normal conformal connexion of C_n fixes the hyperspheres $P-Q$ tangent to the hypersurfaces $p/q=$consts., and hence from the Theorem III.III., the hypersurfaces $p/q=$consts., are all totally umbilical. Moreover, the curves whose directions are always orthogonal to $P-Q$ are, according to the Theorem II.III., conformal circles, hence we have the

Theorem V.I. If the group of holonomy of C_n fixes two points or hyperspheres of the type (B), C_n admits a family of ∞^1 totally umbilical hypersurfaces whose orthogonal trajectories are conformal circles. (S. II. Theorem I.)

We can give, to this theorem of S. Sasaki, a concircular geometrical interpretation. According to the theorems in § 1, the C_n is conformal to two Einstein spaces which are not mapped by a trivial mapping, that is to say, not by a conformal mapping of the form $\bar g_{\mu\nu}=k^2g_{\mu\nu}$ where k is a constant, hence, these Einstein spaces are mapped conformally to each other by a non-trivial mapping. Consequently, as stated in "Concircular geometry V", the C_n is conformal to an Einstein space which admits concircular transformations, and we have the

Theorem V.II. The C_n whose group of holonomy fixes two points or hyperspheres is conformal to an Einstein space which admits concircular transformations.

According to the Theorem IV in "Concircular geometry II", the Theorem V.I is only a corollary of this theorem.

If an Einstein space with the scalar curvature R is non trivially conformal to an another Einstein space with the scalar curvature $\bar R$, then the partial differential equations

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1) K. Yano: Concircular geometry II, loc. cit.

$$(5.1) \qquad \rho_{\mu;\nu}-\rho_\mu\rho_\nu+\frac{1}{2}g^{\alpha\beta}\rho_\alpha\rho_\beta=\frac{1}{2n(n-1)}(R-\bar{R}\rho^2)g_{\mu\nu}$$

obtained from $\bar{\Pi}^0_{\mu\nu}=\Pi^0_{\mu\nu}+\rho_{\mu\nu}$ must be completely integrable. The necessary and sufficient condition that it may be so is that the space admits a family of ∞^1 totally umbilical hypersurfaces whose orthogonal trajectories are geodesic Ricci curves. But this condition does not depend on the constant $\bar{R}$, hence we have the

Theorem V.III. If an Einstein space with scalar curvature R is non trivially conformal to an another Einstein space with the scalar curvature $\bar{R}$, then the original Einstein space is also non trivially conformal to an Einstein space with any scalar curvature $\bar{\bar{R}}$. (Cf. S. I. Theorem 3, 4, 5, 6.)

Now, on a totally umbilical hypersurface in C_n whose group of holonomy fixes the point A_∞ or the hypersphere $-cA_0+A_\infty$, we have from (3.1),

$$\left(\frac{1}{2}H^2-c\right)A_{\dot{0}}+A_{\dot{\infty}}=HA_{\dot{n}}-cA_0+A_\infty,$$

where $H=\frac{1}{n-1}H^{a}_{\cdot a\dot{n}}$. Hence the formulae (3.4) and $d(-cA_0+A_\infty)=0$ show that

$$d\left[\left(\frac{1}{2}H^2-c\right)A_{\dot{0}}+A_{\dot{\infty}}\right]=0 \quad \text{and} \quad H=\text{const.},$$

and give the

Theorem V.IV. If there exists a totally umbilical hypersurface in a C_n whose group of holonomy fixes a point or a hypersphere, the group of holonomy of the induced connexion of the hypersurface fixes also a point or a hypersphere and the mean curvature of the hypersurface is a constant. (S. II. Theorem 2.)

§ 6. *Conformal and concircular geometries in Einstein spaces.*

In "Concircular geometry V.", we have proved that:

The necessary and sufficient condition that an Einstein space whose scalar curvature R is positive, zero or negative admits a concircular transformation is that the fundamental quadratic differential form may be respectively reduced to the followings:

Case I. $K=\frac{1}{n(n-1)}R>0.$

$$(6.1) \quad ds^2=(A\cos\sqrt{K}x^n+B\sin\sqrt{K}x^n)^2\,{}^*g_{jk}(x^i)dx^jdx^k+dx^ndx^n,$$

the Riemannian space ${}^*V_{n-1}$ whose fundamental quadratic differential form is ${}^*g_{jk}dx^jdx^k$ being also an Einstein space with positive scalar curvature ${}^*R=\frac{n-2}{n}(A^2+B^2)R.$

Case II. $K=\frac{1}{n(n-1)}R=0.$

$$(6.2) \qquad ds^2=(Ax^n+B)^2\,{}^*g_{jk}(x^i)dx^jdx^k+dx^ndx^n,$$

the Riemannian space ${}^*V_{n-1}$ whose fundamental quadratic differential form is ${}^*g_{jk}dx^jdx^k$ being also an Einstein space with positive scalar curvature ${}^*R=(n-1)(n-2)A^2$.

Case III. $K=\frac{1}{n(n-1)}R<0$.

$$(6.3)\qquad ds^2=(Ae^{\sqrt{-K}x^n}+Be^{-\sqrt{-K}x^n})^2\,{}^*g_{jk}(x^i)dx^jdx^k+dx^ndx^n\,,$$

the Riemannian space ${}^*V_{n-1}$ whose fundamental quadratic differential form is ${}^*g_{jk}dx^jdx^k$ being also an Einstein space with scalar curvature ${}^*R=\frac{4(n-2)}{n}ABR$.

On the other hand, S. Sasaki has proved that:

If the group of holonomy of a space with a normal conformal connexion fixes two points or hyperspheres, the line-element of the space can be reduced to the canonical form

$$(6.4)\qquad ds^2=[(x^n)^2+k]^2\,{}^*g_{jk}(x^i)dx^jdx^k+dx^ndx^n\,,$$

where k is a constant $>$, $=$, <0 according as the pencil of hyperspheres determined by the invariant hyperspheres is hyperbolic, parabolic or elliptic, and the Riemann space ${}^*V_{n-1}$ with the fundamental quadratic differential form ${}^*g_{jk}dx^jdx^k$ is an Einstein space. The converse is also true.

The line-element (6.4) given by S. Sasaki is not the line-element of an Einstein space, but it is the line-element of a space conformal to an Einstein space, while our line-elements (6.1), (6.2) and (6.3) are those of an Einstein space and must be conformal to (6.4) given by S. Sasaki.

As a matter of fact, multiplying (6.1) by $(A\cos\sqrt{K}x^n+B\sin\sqrt{K}x^n)^2$ and putting $\bar{x}^n=\frac{1}{\sqrt{K}}(A\sin\sqrt{K}x^n-B\cos\sqrt{K}x^n)$, we have

$$(6.5)\qquad d\bar{s}^2=\left[(\bar{x}^n)^2-\frac{A^2+B^2}{K}\right]^2{}^*g_{jk}(x^i)dx^jdx^k+d\bar{x}^nd\bar{x}^n\,,$$

which corresponds to the case $k=-\frac{A^2+B^2}{K}<0$, multiplying (6.2) by $\frac{A^4}{(Ax^n+B)^2[\log(Ax^n+B)]^4}$ and putting $\bar{x}^n=-\frac{A}{\log(Ax^n+B)}$, we have

$$(6.5)\qquad d\bar{s}^2=(\bar{x}^n)^4\,{}^*g_{jk}(x^i)dx^jdx^k+d\bar{x}^nd\bar{x}^n\,,$$

which corresponds to the case $k=0$, and finally multiplying (6.3) by $(Ae^{\sqrt{-K}x^n}+Be^{-\sqrt{-K}x^n})^2$ and putting $\bar{x}^n=\frac{1}{\sqrt{-K}}(Ae^{\sqrt{-K}x^n}-Be^{-\sqrt{-K}x^n})$, we have

$$(6.6)\qquad d\bar{s}^2=\left[(\bar{x}^n)^2-\frac{4AB}{K}\right]^2{}^*g_{jk}(x^i)dx^jdx^k+d\bar{x}^nd\bar{x}^n\,,$$

which corresponds to the case $k>0$, $=0$ or <0 according as $AB>0$, $AB=0$ or $AB<0$.