88. *On Non-prolongable Riemann Surfaces.*

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Let F be a Riemann surface spread over the w-plane. T. Radó[1] called F prolongable, if we can map F on a proper part $\mathfrak{F}_0$ of another Riemann surface $\mathfrak{F}$ spread over the z-plane and non-prolongable, if otherwise. A closed Riemann surface is evidently non-prolongable, but Radó proved that there exists an open non-prolongable Riemann surface by an example of a Riemann surface, which consists of two sheets, whose branch points lie at points $w=n\ (n=0, \pm 1, \pm 2, \dots)$. We will give a class of open non-prolongable Riemann surfaces, which contains the above Radó's example as a special case.

Theorem. Let F be a Riemann surface spread over the w-plane, which consists of n sheets and whose every boundary point is a cluster point of branch points and such that the set E of projections of boundary points of F on the w-plane is a closed set of capacity[2] *zero. Then F is non-prolongable.*

Proof. Suppose that F is prolongable and that we can map F by $w=f(z)$ on a proper part $\mathfrak{F}_0$ of another Riemann surface $\mathfrak{F}$ spread over the z-plane. Then there exists an inner point z_0 of $\mathfrak{F}$, which is a boundary point of $\mathfrak{F}_0$. We may assume that z_0 is not a branch point of $\mathfrak{F}$, since otherwise, we can map $\mathfrak{F}$ by $(z-z_0)^{\frac{1}{p}}$ on another Riemann surface, such that z_0 corresponds to an inner point differing from the branch point.

From the definition of E, there exists at least one boundary point of F above any point P of E, but there may exist inner points of F above P.

We take off all points from $\mathfrak{F}_0$, which are the images of inner points of F lying above E and the remaining part of $\mathfrak{F}_0$ be denoted by $\mathfrak{F}_0'$. We take ρ so small that all points of a disc: $|z-z_0|\leqq\rho$ are inner points of $\mathfrak{F}$ differing from branch points.

Let $\mathfrak{F}_0(\rho)$, $\mathfrak{F}_0'(\rho)$ be the part of $\mathfrak{F}_0$, $\mathfrak{F}_0'$ inside a circle C: $|z-z_0|=\rho$ and $e_0(\rho)$, $e_0'(\rho)$ be the sets of boundary points of $\mathfrak{F}_0(\rho)$, $\mathfrak{F}_0'(\rho)$ inside C respectively.

Since cap. $E=0$, we see that $e_0'(\rho)$ differs from $e_0(\rho)$ only by a set of capacity zero. Now in $\mathfrak{F}_0'(\rho)$, $f(z)$ does not take values belonging to E and since F consists of only n sheets, if z tends to $e_0'(\rho)$, then $w=f(z)$ has cluster points belonging to E. Hence by a lemma[3] proved before, we have cap. $e_0'(\rho)=0$ and hence cap. $e_0(\rho)=0$. $w=f(z)$ is one-

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1) T. Radó: Über eine nicht fortsetzbare Riemannsche Mannigfaltigkeit. Math. Zeits. **20** (1924).

2) In this paper, "capacity" means "logarithmic capacity".

3) M. Tsuji: On the Riemann surface of an inverse function of a meromorphic function in the neighbourhood of a closed set of capacity zero. Proc. **19** (1943).

valued and meromorphic in the neighbourhood of $e_0(\rho)$ and since every point of E is a cluster point of branch points of F, $f(z)$ has an essential singularity at every point of $e_0(\rho)$. Hence by Nevanlinna-Kametani's theorem[1], $f(z)$ takes any value infinitely many times in the neighbourhood of $e_0(\rho)$, except a set of values of capacity zero, which contradicts the hypothesis, that F consists of only n sheets. Hence F is non-prolongable, q. e. d.

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1) R. Nevanlinna: Eindeutige analytische Funktionen. p. 32. S. Kametani: The exceptional values of functions with the set of capacity zero of essential singularities. Proc. **17** (1941). M. Tsuji: On the behaviour of a meromorphic function in the neighbourhood of a closed set of capacity zero. Proc. **18** (1942).