86. Notes on Banach Space (VII): Compactness of Function Spaces.

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Recently Prof. Izumi[1] has derived from his key theorem the following theorem.

A set $\mathfrak{F}$ in E where E is (C), (L^p, $\infty > p \geqq 1$), etc. is compact when and only when

1° $\|f(x)\| \leqq M$ *for all* $f(x) \in \mathfrak{F}$,

2° $\lim_{\delta\to 0} \frac{1}{\delta}\int_0^\delta f(x+t)dt = f(x)$ *uniformly in* $\mathfrak{F}$.

These conditions are of the Kolmogoroff[2]-Tulajkov[3] type. But there are conditions of the Arzèla-M. Riesz[4] type. In the present note the author establishes an abstract theorem of the latter type.

Theorem. Let E be a Banach space satisfying $\lim_{t\to 0}\|f(x+t)-f(x)\|=0$. *If* 1° *and* 2° *are compactness conditions of a set* $\mathfrak{F}$ *in* E, *then they are equivalent to the following*

1° $\|f(x)\| \leqq M$ *for all* $f(x) \in \mathfrak{F}$,

2^{∞} $\lim_{t\to 0}\|f(x+t)-f(x)\|=0$ *uniformly in* $\mathfrak{F}$.

Proof. Necessity. 1° is evident. If $\mathfrak{F}$ is compact, then it is totally bounded. So for any $e > 0$, there are $f_1, f_2, \dots, f_n$ in $\mathfrak{F}$ such that for any $f \in \mathfrak{F}$ there is a k such as $\|f-f_k\| < e$. Since $\lim_{t\to 0}\|f(x+t)-f(x)\|=0$, we have

$$\begin{aligned}\|f(x+t)-f(x)\| &= \|f(x+t)-f_k(x+t)+f_k(x+t)-f_k(x)+f_k(x)-f(x)\| \\ &\leqq \|f(x+t)-f_k(x+t)\|+\|f_k(x+t)-f_k(x)\|+\|f_k(x)-f(x)\| \\ &\leqq 3e.\end{aligned}$$

Thus we get 2^{∞}, and then the necessity of the condition.

Sufficiency. We suppose that $f(x+t)-f(x)$ is an abstract function of t whose range lies in E. Then the function is measurable in the Bochner sense[5]. For any $e > 0$, there is a $\delta=\delta(e)$ such that $\|f(x+t)-f(x)\| < e$ ($|t| < \delta$). Therefore $f(x+t)-f(x)$ is bounded and then it is integrable in the Bochner sense.

By 2^{∞},

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1) S. Izumi, Proc. **19** (1943), 99-101.
2) A. Kolmogoroff, Göttinger Nachrichten, (1931), 60-63.
3) A. Tulajkov, ibid., (1933), 167-170.
4) M. Riesz, Acta Szeged, **6** (1932-34), 136-142.
5) S. Bochner, Fund. Math., **20** (1933), 262-276.

$$\left\| \frac{1}{\delta}\int_0^\delta f(x+t)dt - f(x) \right\| = \frac{1}{\delta} \left\| \int_0^\delta \big(f(x+t)-f(x)\big)dt \right\|$$

$$\leqq \frac{1}{\delta}\int_0^\delta \|f(x+t)-f(x)\|\, dt \leqq e$$

uniformly in $\mathfrak{F}$. Thus we get 2° and then $\mathfrak{F}$ is compact from Izumi's theorem.