2. *On Conformal Mapping of an Infinitely Multiply Connected Domain.*

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1. Let G be a Fuchsian group of linear transformations, which make $|z|<1$ invariant and D_0 be its fundamental domain containing $z=0$ and bounded by orthogonal circles to $|z|=1$ and D_n be its equivalent and e_n be the set on $|z|=1$, which belongs to the boundary of D_n. Let z_0 be a point in D_0 and z_n be its equivalent in D_n.

Theorem I. *If* $me_0>0$, *then* $\sum_{n=0}^{\infty} me_n=2\pi$ *and* $\sum_{n=0}^{\infty}(1-|z_n|)<0$.

$$\text{If } me_0=0, \text{ then } \sum_{n=0}^{\infty} me_n=0 \text{ and } \sum_{n=0}^{\infty}(1-|z_n|)=\infty\,,$$

$$\sum_{n=0}^{\infty}(1-|z_n|)^2<\infty\,.$$

Let D be a domain on the w-plane, bounded by a closed set E, which contains at least three points and $\mathfrak{F}^{(\infty)}$ be the simply connected universal covering Riemann surface of the outside of E. We map $\mathfrak{F}^{(\infty)}$ on $|z|<1$ by $w=\varphi(z)$. R. Nevanlinna[1] proved that if cap. $E>0$, then E corresponds to a set of measure 2π on $|z|=1$ and if cap. $E=0$, then E corresponds to a set of measure zero on $|z|=1$, when z tends to $|z|=1$ non-tangentially. $\varphi(z)$ is automorphic with respect to a group G of linear transformations, which make $|z|<1$ invariant. Let D_0 be its fundamental domain containing $z=0$ and bounded by orthogonal circles to $|z|=1$ and D_n be its equivalent and e_n be the set on $|z|=1$, which belongs to the boundary of D_n. Then from Theorem I, we have easily:

Theorem II (*Precised form of R. Nevanlinna's theorem*).

$$\text{If cap. } E>0, \text{ then } \sum_{n=0}^{\infty} me_n=2\pi\,.$$

$$\text{If cap. } E=0, \text{ then } \sum_{n=0}^{\infty} me_n=0$$

2. Let F be a Riemann surface spread over the w-plane and $F^{(\infty)}$ be its covering Riemann surface of planar character and $\mathfrak{F}^{(\infty)}$ be its simply connected universal covering Riemann surface. We map $F^{(\infty)}$ on a schlicht domain D on the z-plane. D is the outside of a certain closed set E. We suppose that we can map $\mathfrak{F}^{(\infty)}$ on a unit circle $|\zeta|<1$ by $w=\varphi(\zeta)$. $\varphi(\zeta)$ is automorphic with respect to a group G of linear transformations, which make $|\zeta|<1$ invariant. Let D_0 be its fundamental domain containing $\zeta=0$ and bounded by orthogonal

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1) R. Nevanlinna: Eindeutige analytische Funktionen. Berlin, 1936.

circles to $|\zeta|=1$ and e_0 be the set on $|\zeta|=1$, which belongs to the boundary of D_0. Then

Theorem III (Fundamental theorem). cap. $E>0$, when and only when $me_0>0$.

3. Let F be a Riemann surface spread over the w-plane. Green's function $G(w, w_0)$ of F is defined as follows. We approximate F by a sequence of Riemann surfaces: $F_1 \subset F_2 \subset \cdots \subset F_n \to F$, where F_n contains w_0 and is bounded by a finite number of closed curves on F and consists of only inner points of F. Let $G_n(w, w_0)$ be Green's function of F_n with w_0 as its pole. By Harnack's theorem, $\lim_{n\to\infty} G_n(w, w_0)=G(w, w_0)$ uniformly on F, where $G(w, w_0)\equiv\infty$ or is a harmonic function on F, except at w_0, where it has a logarithmic singularity. If $G(w, w_0)\not\equiv\infty$, we call it Green's function of F. Let $\mathfrak{F}^{(\infty)}$ be the simply connected universal covering Riemann surface of F. We suppose that we can map $\mathfrak{F}^{(\infty)}$ on $|\zeta|<1$ by $w=\varphi(\zeta)$. $\varphi(\zeta)$ is automorphic with respect to a group G of linear transformations, which make $|\zeta|<1$ invariant. Let D_0 be its fundamental domain containing $\zeta=0$ and bounded by orthogonal circles to $|\zeta|=1$ and e_0 be the set on $|\zeta|=1$, which belongs to the boundary of D_0. Then

Theorem IV. Green's function of F exists, when and only when $me_0>0$.

Myrberg[1] proved that if there exists a non-constant positive harmonic function on F, then Green's function of F exists. We can prove:

Theorem V. If Green's function of F exists, then there exists a non-constant positive bounded harmonic function on F.

4. Let $G(x, y)$ be an integral function with respect to x and y and $y=y(x)$ be an analytic function defined by $G(x, y)=0$ and F be its Riemann surface spread over the x-plane. In the former paper[2], I have proved that if $y(x)$ is not an algebroid function, then F covers any point infinitely many times, except a set of points of capacity zero and the set of projections of direct transcendental singularities of $y(x)$ on the x-plane is of capacity zero. Let $F^{(\infty)}$ be the covering Riemann surface of F of planar character. We map $F^{(\infty)}$ on a schlicht domain D on the z-plane by $x=f(z)$. D is the outside of a certain closed set E. $f(z)$ is automorphic with respect to a group G of transformations $z'=U(z)$, which transforms the outside of E into itself. Let $\mathfrak{F}^{(\infty)}$ be the simply connected universal covering Riemann surface of F. We suppose that we can map $\mathfrak{F}^{(\infty)}$ on a unit circle $|\zeta|<1$ by $x=\varphi(\zeta)$. $\varphi(\zeta)$ is automorphic with respect to a group $\bar{G}$ of linear transformations, which make $|\zeta|<1$ invariant. Let D_0 be its fundamental domain containing $\zeta=0$ and bounded by orthogonal circles to $|\zeta|=1$ and e_0 be the set on $|\zeta|=1$, which belongs to the boundary of D_0. If $me_0>0$, then we can easily prove that almost all points of e_0 correspond to the boundary points of F. Now the boundary of

1) Myrberg: Über die Existenz der Greenschen Funktionen auf einer gegebenen Riemannschen Fläche. Acta Math. **61**.

2) M. Tsuji: On the domain of existence of an implicit function defined by an integral relation $G(x, y)=0$. Proc. **19** (1943),

F consists of a point $x=\infty$ and points x such that $y(x)=\infty$, so that by Lusin-Priwaloff's theorem[1], $x=\varphi(\zeta)\equiv\infty$ or $y(\varphi(\zeta))\equiv\infty$, which is impossible. Hence $me_0=0$, so that by Theorem III, we have cap. $E=0$. From cap. $E=0$, we can prove[2] that $U(z)$ is a linear function of z. Thus we have

Theorem VI. A curve $G(x, y)=0$ can be uniformized by automorphic functions belonging to a linear group of Schottky type, whose singular set is of capacity zero.

Since $F^{(\infty)}$ is the Riemann surface of the inverse function of $x=f(z)$, which is one-valued and meromorphic outside a closed set E of capacity zero, we have[3]

Theorem VII (Extension of Gross' theorem). Let $y=y(x)$ be defined by $G(x, y)=0$ and x_0 be a regular point of $y(x)$. Then $y(x)$ can be continued analytically on half-lines $x=x_0+re^{i\theta}(0\leqq r<\infty)$ indefinitely, except for θ-values of measure zero.

5. We have the following

Theorem VIII (Extension of Lusin-Priwaloff's theorem). Let E be a closed set of capacity zero on the w-plane and e be a set of positive measure on $|z|=1$ and $w=f(z)$ be meromorphic in $|z|<1$. If $\lim f(z)$ exists, when z tends to e non-tangentially to $|z|=1$ and the limiting values belong to E, then $f(z)\equiv$const.

From this we have

Theorem IX (Extension of R. Nevanlinna's theorem). Let $w=f(z)$ ($\not\equiv$const) be meromorphic in $|z|<1$ and e be a set of positive measure on $|z|=1$. Then the cluster set of $f(z)$ on e, when z tends to e non-tangentially to $|z|=1$, is of capacity positive.

R. Nevanlinna[4] proved under the condition, that the characteristic function $T(r)$ of $f(z)$ is bounded in $|z|<1$.

From Theorem IX, we can prove:

Theorem X. Let E be a closed set of positive capacity on the w-plane and $w=f(z)$ be one-valued and meromorphic in a neighbourhood U of E. Let $z_0\in U-E$ and E_ρ be the sub-set of E, which lies in $|z-z_0|<\rho$ and of positive capacity. Then the cluster set of $f(z)$ on E_ρ is of capacity positive.

6. By Theorem III, we can prove:

Theorem XI. Let D be a domain on the w-plane, bounded by enumerably infinite number of continua $K_i(i=1, 2, \dots)$ and a closed set E of capacity zero, to which different continua cluster, where E may have common points with K_i. Then D can be mapped conformally on a domain bounded by enumerably infinite number of circles and a closed set of capacity zero.

The problem of conformal mapping of an infinitely multiply

1) Lusin-Priwaloff. Sur l'unicité et multiplicité des fonctions analytiques. Ann. Sec. norm. sup. **42** (1925).

2) M. Tsuji: Theory of conformal mapping of a multiply connected domain. Jap. Jour. Math. **18** (1942).

3) M. Tsuji: On the behaviour of a meromorphic function in the neighbourhood of a closed set of capacity zero. Proc. **18** (1942).

4) R. Nevanlinna. l. c. 1).

connected domain on a domain bounded by circles was proposed by Koebe[1] in the congress at Rome in 1908 as desideratum. From Lichtenstein's article: Neuere Entwicklung der Potentialtheorie. Konforme Abbildung in the Enzyklopädie der mathematischen Wissenschaften, II, we know only special cases are solved till now.

Theorem XII. Let D be a domain on the w-plane, bounded by enumerably infinite number of circles $C_i(i=1, 2, \dots)$ and a closed set E of capacity zero, where E may have common points with C_i and C_i may touch each other externally. We invert D into one of C_i and performing the similar operations on all circles and circles newly obtained, we obtain infinitely many circles clustering to a closed set M. Then M is of capacity zero.

From this we can prove:

Theorem XIII. Let D be a domain on the z-plane of the nature mentioned in Theorem XII and Δ be a domain on the w-plane of the same nature. If we can map D conformally on Δ by $w=f(z)$, then $f(z)$ is a linear function of z.

The full detail of the proof will apear in Japanese Journal of Mathematics, **19** (1944).

1) Koebe: Über ein allgemeines Uniformisierungsprinzip. Atti del congresso intern. dei. Mat. Roma, **2** (1909).