16. *On Group Rings of Topological Groups.*

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§ 1. Let G be a locally compact topological group, satisfying the second axiom of countability and μ a left invariant Haar measure on G. We denote as usual by $L^p(G)$ $(p \geqq 1)$ the set of all μ-measurable functions $x(g)$ of G with finite

$$\| x(g) \|_p = \left\{ \int_G | x(g) |^p \mu(dg) \right\}^{\frac{1}{p}} .$$

For arbitrary $x(g) \in L^1(G)$, $y(g) \in L^p(G)$ and

$$(1) \qquad z(g) = x \times y(g) = \int_G x(h) y(h^{-1}g) \mu(dh) ,$$

we have

$$(2) \qquad \| z \|_p = \| x \times y \|_p \leqq \| x \|_1 \| y \|_p .$$

Defining the multiplication by (1) and putting

$$(3) \qquad \| x \| = \text{Max.} (\| x \|_1, \ \| x \|_p) ,$$

the intersection $L^{(1,p)}(G)$ of $L^1(G)$ and $L^p(G)$ thus becomes a non-commutative normed ring[1]. But, generally speaking, $L^{(1,p)}(G)$ has not a unit element. Adjoining therefore formally the unit e, I. E. Segal considered the set of all

$$\mathfrak{z} = \lambda e + x(g) ; \quad \lambda = \text{complex number}, \quad x(g) \in L^{(1,p)}(G) ,$$

and called it the group ring $R^{(1,p)}(G)$ of G[2]. But we would rather prefer to call $L^{(1,p)}(G)$ itself the group ring of G. We shall give in this paper certain close relations between G and $L^{(1,p)}(G)$, some of which are generalizations of the results of I. E. Segal.

§ 2. We consider representations of G and $L^{(1,p)}(G)$, i. e. homomorphic mappings of G and $L^{(1,p)}(G)$ into matrices, whose components are complex numbers[3].

Our main theorem is then:

Theorem 1. There is a one-to-one correspondence between continuous[4] representations of $L^{(1,p)}(G)$ and bounded continuous representations of G in the following sense:

i) For a given continuous representation $x(g) \to T(x)$ of $L^{(1,p)}(G)$, there corresponds uniquely a bounded continuous representation $a \to D(a)$ of G, so that it holds

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1) For normed rings cf. I. Gelfand: Normierte Ringe, Rec. Math., **51** (1941), 37–58.

2) I. E. Segal: The group ring of a locally compact group, I, Proc. Nat. Acad. Sci., U. S. A. **27** (1940).

3) For the representation of G, we do not require that the unit of G corresponds to the unit matrix.

4) The topology in $L^{(1,p)}(G)$ is of course given by the norm $\| x \|$ in (3).

$$T(x)=\int_G x(g)D(g)\mu(dg)^{5)} \tag{4}$$

for all $x(g)$ in $L^{(1,p)}(G)$. We denote this representation of G by $D_T(a)$.

ii) Conversely, if $a\rightarrow D(a)$ is an arbitrary bounded continuous representation of G and if we define $T(x)$ by (4), the mapping $x(g)\rightarrow T(x)$ gives a continuous representation for $x(g)$ in $L^{(1,p)}(G)$. We denote this representation by $T_D(x)$.

iii) i) and ii) give mutually inverse correspondences: if $D_1=D_{T_1}$, then $T_1=T_{D_1}$ and if $T_2=T_{D_2}$, then $D_2=D_{T_2}$.

iv) Equivalent representations correspond to each other: if $AT_1(x)A^{-1}=T_2(x)$, then $AD_{T_1}(a)A^{-1}=D_{T_2}(a)$ and if $BD_1(a)B^{-1}=D_2(a)$, then $BT_{D_1}(x)B^{-1}=T_{D_2}(x)$.

From Theorem 1 follow immediately some corollaries: Let $a\rightarrow D(a)$ be a bounded measurable representation of G. If we put

$$T(x)=\int_G x(g)D(g)\mu(dg)\,,$$

$x(g)\rightarrow T(x)$ gives, as before, a continuous representation of $L^{(1,p)}(G)$. By Theorem 1 we have thus $T(x)=T_{D_1}(x)$ with some bounded continuous representation $D_1(a)$ of G and hence $D(a)=D_1(a)$. Thus

Theorem 2. Any bounded measurable representation of G is continuous.

In a similar way we obtain by a simple calculation the following

Theorem 3. If G is locally compact, but not compact, then there is no representation of G belonging to $L^p(G)$ $(p\geqq 1)$ except the zero representation, which maps every element of G to the zero matrix. On the other hand if G is compact, any representation belonging to $L^p(G)$ $(p\geqq 1)$ is bounded and continuous.

Now, as a bounded representation of G is always completely reducible, it follows from Theorem 1, iv) that a continuous representation of $L^{(1,p)}(G)$ is completely reducible. But the converse is also true. It holds namely

Theorem 4. A representation $T(x)$ of $L^{(1,p)}(G)$ is continuous if and only if it is completely reducible. Especially a irreducible representation of $L^{(1,p)}(G)$ is always continuous.

This theorem is equivalent to the following

Theorem 5. Let M be a two-sided ideal of $L^{(1,p)}(G)$, such that the rest class ring $L^{(1,p)}/M$ is of finite dimension. $L^{(1,p)}/M$ is then semi-simple if and only if M is closed in $L^{(1,p)}(G)$. Especially a maximal ideal M is always closed in $L^{(1,p)}(G)$.

These theorems can be regarded as a generalization of the complete reducibility of the group ring of a finite group.

The above mentioned relation between ideals and representations of $L^{(1,p)}(G)$ is explicitly given by

Theorem 6. Let $\{D\}$ be a class of equivalent bounded continuous

5) The right-hand side means a matrix with (i,j)-component $\int_G x(g)d_{ij}(g)\mu(dg)$, where $D(a)=\{d_{ij}(a)\}$.

irreducible representations of G and $D(a)$ be a representant of it. Then all functions $x(g)$ of $L^{(1,p)}(G)$ satisfying

$$\int_G x(g)D(g)\mu(dg)=0\,,$$

constitute a maximal two-sided ideal $M_{\{D\}}$ in $L^{(1,p)}(G)$. $\{D\}\to M_{\{D\}}$ thus gives a one-to-one correspondence between all classes of equivalent bounded continuous irreducible representations of G and all maximal two-sided ideals M of $L^{(1,p)}(G)$, for which $L^{(1,p)}/M$ is of finite dimension.

If we define the classes of (not necessarily irreducible) representations of G suitably, then the result of Theorem 6 can be extended to those classes of representations of G and all closed two-sided ideals M of $L^{(1,p)}(G)$, for which $L^{(1,p)}/M$ is of finite dimension. It follows then also, that for $L^{(1,p)}(G)$, $p=1, 2, \dots$ the ideals of that kind correspond one-to-one to each other.

§ 3. In order to establish corresponding theorems for Segal's group ring $R^{(1,p)}(G)$, we have only to prove the following

Lemma. Let M be a two-sided ideal in $L^{(1,p)}(G)$ such that the rest class ring $L^{(1,p)}/M$ is of finite dimension and has a unit element. Then a two-sided ideal M' of $R^{(1,p)}(G)$ can be uniquely determined, so that it holds

$$M=M'\frown L^{(1,p)}(G)\,, \qquad R^{(1,p)}(G)=M'\smile L^{(1,p)}(G)\,.$$

From this lemma it follows that closed ideals in $R^{(1,p)}(G)$ which are not contained in $L^{(1,p)}(G)$ and closed ideals in $L^{(1,p)}(G)$ correspond one-to-one to each other. Making use of this fact and Theorem 1 we obtain

Theorem 7. For a continuous representation[6] $\mathfrak{z}\to T(\mathfrak{z})$ of $R^{(1,p)}(G)$ there is a continuous bounded representation $a\to D(a)$ of G, so that for any $\mathfrak{z}=\lambda e+x(g)$ in $R^{(1,p)}(G)$ it holds

$$T(\mathfrak{z})=T\big(\lambda e+x(g)\big)=\lambda D(1)+\int_G x(g)D(g)\mu(dg)^{7)}\,. \tag{5}$$

Conversely, for any continuous bounded representation $a\to D(a)$ of G, $T(\mathfrak{z})$ in (5) gives a continuous representation $\mathfrak{z}\to T(\mathfrak{z})$ of $R^{(1,p)}(G)$ and thus continuous representations of $R^{(1,p)}(G)$ and continuous bounded representations of G correspond one-to-one to each other.

We can also prove similar theorems to Theorems 4, 5, 6. Especially a one-to-one correspondence is to be established between all classes of irreducible bounded continuous representations of G and all maximal ideals $M'\neq L^{(1,p)}(G)$ of $R^{(1,p)}(G)$, for which $R^{(1,p)}/M'$ is of finite dimensions[8].

§ 4. We now extend our Theorem 1 to representations of G and $L^{(1,p)}(G)$ by bounded operators in a Hilbert space $\mathfrak{H}$. Let $\boldsymbol{B}$ be the

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6) We define the norm in $R^{(1,p)}(G)$ by $\|\mathfrak{z}\|=\|\lambda e+x(g)\|=|\lambda|+\|x\|$ and say "continuous" in the sense of this norm.

7) $D(1)$ is the matrix corresponding to the unit of G.

8) Cf. Segal, l. c. 2).

ring of all bounded operators in $\mathfrak{H}$. A representation $a \to D(a)\big(D(a) \in \boldsymbol{B}\big)$ of G in $\boldsymbol{B}$ is called a "proper" representation, when it holds $D(1)=E$, where 1 means the unit in G and E is the unit operator in $\boldsymbol{B}$. It is called bounden, if there exists a constant C so that

$$|||\, D(a)\, ||| \leqq C, \qquad \text{for all} \quad a \in G^{9)}.$$

On the other hand we call a representation $x(g) \to T(x)\big(T(x) \in \boldsymbol{B}\big)$ of $L^{(1,p)}(G)$ in $\boldsymbol{B}$ "proper", if $T(x)f=0(f \in \mathfrak{H})$ for all $x(g) \in L^{(1,p)}(G)$ implies $f=0$, and we call it "continuous", if there is a constant C' so that

$$|||\, T(x)\, ||| \leqq C' \, \| x \|, \qquad \text{for all} \quad x(g) \in L^{(1,p)}(G)^{10)}.$$

We can now prove the following

Theorem 8. There is a one-to-one correspondence between continuous proper representations of $L^{(1,p)}(G)$ in $\boldsymbol{B}$ and bounded measurable[11] proper representations of G in $\boldsymbol{B}$ in the following sense:

i) For such a representation $x(g) \to T(x)$ of $L^{(1,p)}(G)$ in $\boldsymbol{B}$, there is a bounded measurable proper representation $a \to D(a)$ of G, so that

$$T(x) = \int_G x(g) D(g) \mu(dg)^{12)} \tag{6}$$

for all $x(g)$ in $L^{(1,p)}(G)$. Such $D(a)$ is uniquely determined by $T(x)$.

ii) Conversely, if $a \to D(a)$ is such a representation of G, $T(x)$ in (6) gives a continuous proper representation of $L^{(1,p)}(G)$ in $\boldsymbol{B}$.

iii) Above correspondences are mutually inverse.

iv) Equivalent representations correspond to each other[13].

If the measure μ is not only left invariant, but also right invariant, then we can obtain some more precise results. We can thus prove for example the following theorem.

Theorem 9. If G has an invariant Haar measure, then measurable unitary representations of G in $\boldsymbol{B}$ are all strongly continuous[14].

Detailed proofs of above theorems will appear elsewhere. They need some considerations on non-commutative normed rings[15] and rings of operators in a Hilbert space, as will be also discussed there precisely[16].

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9) $|||\, A\, |||$ means the bound of the operator A.

10) Thus we consider in $\boldsymbol{B}$ the uniform topology.

11) That is to say, that $(D(g)f, f')$ is μ-measurable for any f, f' in $\mathfrak{H}$.

12) (6) means $(T(x)f, f') = \int_G x(g)\,(D(g)f, f')\mu(dg)$ for any f, f' in $\mathfrak{H}$.

13) Cf. Theorem 1, iii), iv).

14) Cf. K. Kodaira: Über die Gruppen der messbaren Abbildungen, Proc. **17** (1941), 18–23.

15) Some of the theorems, obtained by I. Gelfand, concerning commutative normed rings can be transferred to our non-commutative case.

16) Cf. also author's note in Zenkoku Sijo Sugaku Danwakai, **246** (1942), 1522–1555, **251** (1943), 167–186.