30. On the Phenomena of Instability in Undamped Quasi-harmonic Vibration. Part II.

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1. *Preliminary notes.* In the previous paper[1], some cases of quasi-harmonic vibration, namely, the cases in which there are periodically varying inertia mass as well as elasticity, have been discussed from mathematical point[2]. It has been concluded that instability with a certain range occurs when 1-, 1/3, ... cycle, namely, odd number fraction of a cycle of ripple in periodically varying coefficient synchronizes with two cycles of reference natural vibration, whereas instability with zero range should occur when 1/2-, 1/4-, ... cycle, namely, even number fraction of a cycle of ripple in periodically varying coefficient synchronizes with two cycles of reference natural vibration[3]. In the present paper, results of our experimental investigation with a model as well as the formulation of the equations most adapted to such experiments are mainly stated, from which it is possible for us to ascertain that features found from theory well agree with experimental phenomena.

2. *Experiments with a rotating two-blade model propeller.* We shall now consider such a special case that the propeller and the engine, as a whole, are liable to be in tilting motion under a finite resistance in the elastic force of the engine mounting. In the present case, furthermore, tilting motion in a plane is only present, in consequence of which the experiment and the analysis are much simplified. The skeleton view of the model is shown in Fig. 1.

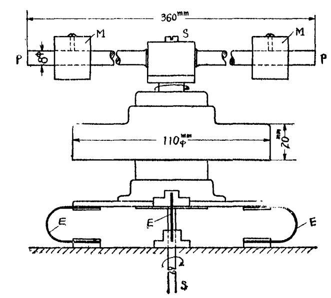

Fig. 1.

A bar PP corresponding to the propeller rotates with a shaft SS. It is possible for the reference natural vibrations to change by shifting the affixed masses M, M along PP. The elastic resistance corresponding to the engine mounting arises from the springs E, E, E. The condition represented in Fig. 1 indicates the state with the reference natural vibration

1) K. Sezawa and I. Utida, Proc. **19** (1943), 646–652.

2) Erratum to the previous paper. $\sqrt{1-k^2\sin\tau}$ in line 3, p. 647 should be read as $\sqrt{1-k^2\sin^2\tau}$.

3) Some careless explanation in the previous paper is now corrected.

(a) "1-, 3-, ... cycles" and "odd numbers of cycles" in line 2, p. 651 should be "1-, 1/3-, ... cycle" and "odd number fraction of a cycle", respectively.

(b) "2-, 4-, ... cycles" and "even numbers of cycles" in lines 5, 6, p. 651, should be "1/2-, 1/4-, ... cycle" and "even number fraction of a cycle", respectively.

(c) "multiple" in line 19, p. 651, is to be "fraction".

of the type ω_0. In such another position of the propeller as turned by 90°, the reference natural vibration is to be ω_1. The shaft SS is connected with a flexible shaft to a one-H. P. three phase Schrage motor and rotated for such a wide range as from 125 rev./min. to 2,500 rev./min. Mass corresponding to part of the propeller is 400 gms. and the mass including the propeller and the engine above the spring, entirely, 3,000 gms. The values of ω_0, ω_1 together with $k'=\omega_0/\omega_1$ are shown in Table I.

TABLE I.

Case	ω_0	ω_1	k'
a	11.45	13.15	0.871
b	10.05	12.0	0.838
c	9.00	12.0	0.750
d	8.35	11.95	0.699

In the present experiment, different cycles ($2p$) of ripple in periodically varying coefficient, namely, the moment of inertia mass, are set up from the change in the revolution in Schrage motor. Taking the value corresponding to ω_1/p as abscissa and the (double) amplitude (in mm) of the propeller tip as ordinate, the experimental results obtained are shown in Figs. 2 a, 2 b,

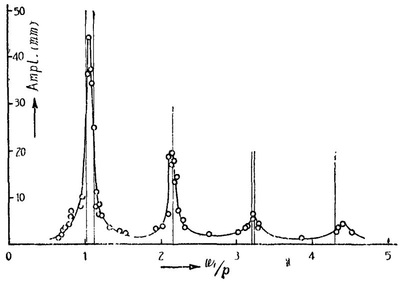

Fig. 2 a. Case a. $k'=0.871$.

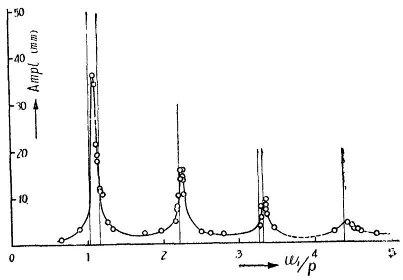

Fig. 2 b. Case b. $k'=0.838$.

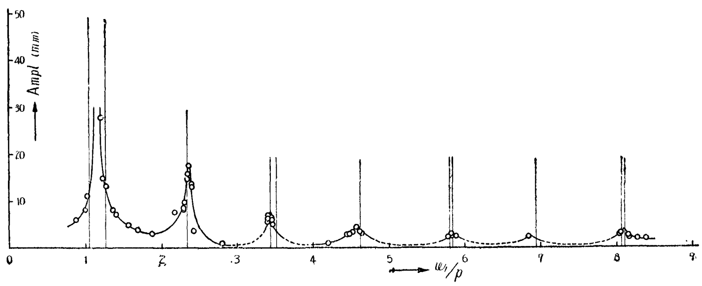

Fig. 2 c. Case c. $k'=0.750$.

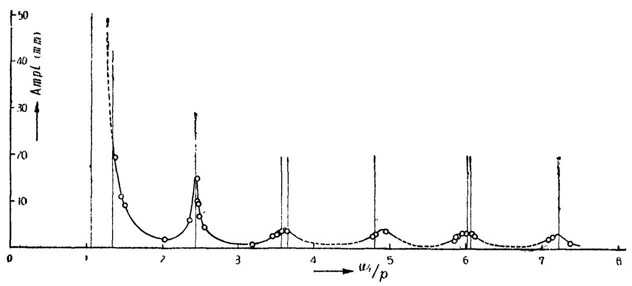

Fig. 2 d. Case d. $k'=0.699$.

2c, 2d. Every maximum in any curve represents the condition of instability. It will be seen that the greater the difference between ω_1 and ω_0, the more pronounced the feature of the reference natural vibration disagreeing with an integral fraction of the cycle of ripple in periodically varying coefficient. It should be borne in mind that although the problem is related to instability, the amplitude at every instability is not infinite owing to the effect of damping as well as change of state at a finite amplitude.

For comparing the experimental result with theoretical one, the values of ω_1/p for every case of ω_0/ω_1 are plotted in Fig. 3, which figure is of the same type as shown in Fig. 1 of the preceding paper. Points plotted in Fig. 3 have been taken from the maximum of every peak in the curves in Figs. 2. For a further confirmation of the answer of the problem, the range of every instability already known

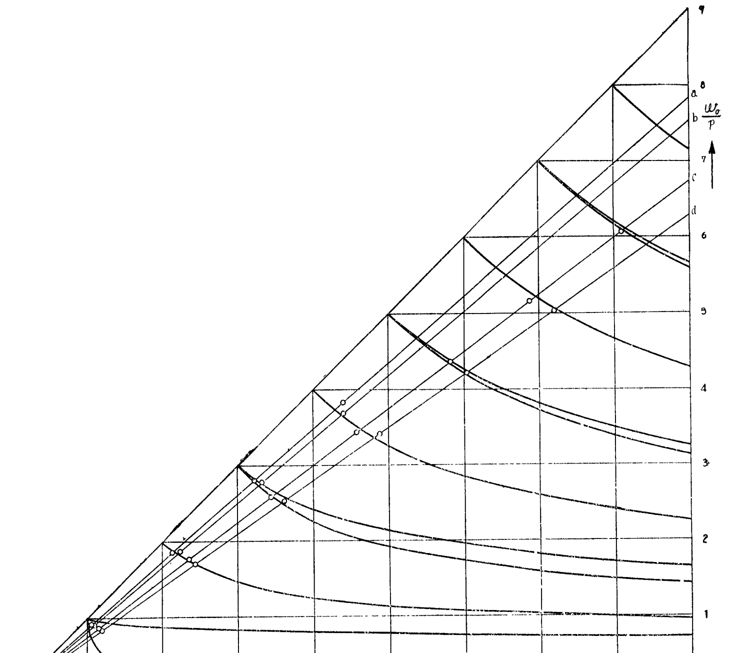

Fig. 3. Instability points in experiments are present nearly in instability ranges known from theory.

from theory is indicated by two vertical strip lines near in every maximum in the experimental results in Figs. 2. Upon examining Figs. 2 as well as Fig. 3, we get the belief that the result from theory and that of experiment fairly agree for every case of the problem and for every order of instability.

3. *Notes on the instability of the system of the engine with the propeller in the case of the elastic mounting being radially symmetrical.* Although, for ascertaining the agreement of the experimental result with theory, the tilting motion of the system of the engine and the propeller in experiments has been constrained to be in a plane, the actual condition is rather such that the distribution of the elastic resistance in the mounting is radially symmetrical with respect to the shaft line of the engine. In such a case, if there are two blades in the propeller, it is also likely that the inertia coefficient is peridically varying, as a result of which the phenomena of such instability as we have so far investigated should naturally be present. The existence of such phenomena has first called Bentley's[1] attention although his problem was not mathematical and the said instability seems to be related merely with the case of fundamental order.

At all events, although in the case of tilting motion of the system lying in any radial plane, the mathematical equations should be invariably simultaneous and the solutions slightly complicated, the important part of the problem differing from our idealized case would be nothing more than the gyroscopic motion of the propeller with any number of blades and it is immaterial whether or not the instability in quasi-harmonic vibration in a two-blade propeller specially contributes to the problem.

4. *Formulation of the equation of motion of the engine with a two-blade propeller.* The formulation of the equation of motion in obedience to the state of the model experiments shown previously is not difficult. Carter[2] gave a simple method of formulating the equation in the case of two degrees of freedom in tilting. In our case, it is possible for the tilting to be restricted in a vertical plane.

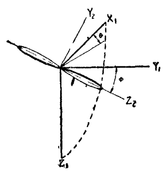

Fig. 4.

Let OX_1, OY_1, OZ_1 in Fig. 4 be fixed rectangular coordinates with O at the centre of the propeller hub. If the engine and propeller system tilts in vertical plane X_1OZ_1 by θ, the angular velocities of the system about OX_1, OY_1, OZ_1, are $0, \dot{\theta}, 0$ respectively. We shall furthermore take axis OZ_2 at the middle line of the blades (generally in any radial line) and axis of OY_2 normal to OZ_2 and OX_1, OZ_2, OY_2 rotating with angular velocity $\dot{\phi}$. It follows that the angular velocities about OX_1, OY_2, OZ_2 are

$$\dot{X}=\dot{\phi}\,, \qquad \dot{Y}=\dot{\theta}\sin\phi\,, \qquad \dot{Z}=\dot{\theta}\cos\phi\,, \tag{1}$$

respectively. Assuming that the moments of inertia of the propeller about OZ_2, OY_2, OX_1 are A, B, C, respectively, the kinetic energy of the propeller about its centre of mass is

$$T_a=\frac{1}{2}(C\dot{X}^2+B\dot{Y}^2+A\dot{Z}^2)\,. \tag{2}$$

Let D_2 be the moment of inertia of the system of the engine and the propeller about the centre of the tilting movement θ, and S_2 the

1) G. P. Bentley, Vibration of Radial Aircraft Engines, J. Aeron. Sci., **6** (1939), No. 7, 283.

2) B. C. Carter, Notes on the Whirling of Radial Engines on their Mountings, Mathematical Analysis and Experimental Observations, R. & M., No. 1783 (1935).

elastic moment of the engine mounting for a unit tilting. The total kinetic energy T and the potential energy V (of the mounting) are then

$$T=\frac{1}{2}(C\dot{X}^2+B\dot{Y}^2+A\dot{Z}^2+D_2\dot{\theta}^2), \qquad V=\frac{S_2}{2}\theta^2. \tag{3}$$

Substituting (3), (1) in Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right)-\frac{\partial T}{\partial\theta}+\frac{\partial V}{\partial\theta}=0, \tag{4}$$

we get

$$\ddot{\theta}(A\cos^2\phi+B\sin^2\phi+D_2)-\dot{\phi}\dot{\theta}(A-B)\sin 2\phi+S_2\theta=0. \tag{5}$$

If the frequency in radian of the propeller rotation be p, then $\phi=pt$, so that (5) reduces to

$$\ddot{\theta}(A\cos^2 pt+B\sin^2 pt+D_2)-\dot{\theta}p(A-B)\sin 2pt+S_2\theta=0, \tag{6}$$

that is to say,

$$\frac{d}{dt}\left\{\left[\left(\frac{A+B}{2}+D_2\right)+\frac{A-B}{2}\cos 2pt\right]\dot{\theta}\right\}+S_2\theta=0. \tag{7}$$

This is of the same form as the equation (1c) in the previous paper[1].

Although we have now discussed the case of a two-blade propeller, the solution is also adapted to cases of any number of blades. If the number of blades were greater than three, it then invariably follows that $A=B$, from which the equation of vibratory motion is simply

$$\{(A+B)/2+D_2\}\ddot{\theta}+S_2\theta=0, \tag{8}$$

no instability being then in existence.

5. *Concluding remarks.* In the present paper, the phenomena of instability in quasi-harmonic vibration have been ascertained experimentally with a two-blade model propeller and furthermore formulated the equation meeting with the case of the propeller for confirming the agreement between theory and experiment from practical standpoint. It is likely that a number of important examples of engineering problems as to relate to the present case is existent, the discussion of which examples, we have hopes, will be available before long.

1) Proc. **19** (1943), 646.