PAPERS COMMUNICATED

27. Construction of a Non-separable Extension of the Lebesgue Measure Space.

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§ 1. A *measure space* $(\Omega, \mathfrak{B}, m)$ is a triple of a space $\Omega=\{\omega\}$, a Borel field $\mathfrak{B}=\{B\}$ of subsets B of Ω, and a countably additive measure $m(B)$ defined on $\mathfrak{B}$ with $0<m(\Omega)<\infty$. In case Ω is the interval $\{\omega \mid 0 \leqq \omega \leqq 1\}$ of real numbers ω, $\mathfrak{B}$ is the Borel field of all Lebesgue measurable subsets B of Ω, and $m(B)$ is the ordinary Lebesgue measure with $m(\Omega)=1$, $(\Omega, \mathfrak{B}, m)$ is called the *Lebesgue measure space.*

For any measure space $(\Omega, \mathfrak{B}, m)$, let $\mathfrak{p}(\Omega, \mathfrak{B}, m)$ be the smallest cardinal number of a subfamily $\mathfrak{A}$ of $\mathfrak{B}$ with the following property: for any $\varepsilon>0$ and for any $B \in \mathfrak{B}$ there exists an $A \in \mathfrak{A}$ such that $m(B \ominus A)<\varepsilon$, where we denote by $B \ominus A$ the symmetric difference $B \cup A-B \cap A$ of B and A. On the other hand, let $L^2(\Omega, \mathfrak{B}, m)$ be the generalized Hilbert space of all real-valued $\mathfrak{B}$-measurable functions $x(\omega)$ defined on Ω which are square integrable on Ω with $\|x\|=\left(\int_\Omega |x(\omega)|^2 m(d\omega)\right)^{\frac{1}{2}}$ as its norm. Then it is easy to see that $\mathfrak{p}(\Omega, \mathfrak{B}, m)$ is equal with the *dimension* of $L^2(\Omega, \mathfrak{B}, m)$ in case the latter is infinite, where we understand by the dimension of $L^2(\Omega, \mathfrak{B}, m)$ the cardinal number of a complete orthonormal system of $L^2(\Omega, \mathfrak{B}, m)$. We shall call $\mathfrak{p}(\Omega, \mathfrak{B}, m)$ the *character* of a measure space $(\Omega, \mathfrak{B}, m)$.

A measure space $(\Omega, \mathfrak{B}, m)$ is *metrically separable* if $\mathfrak{p}(\Omega, \mathfrak{B}, m) \leqq \aleph_0$. This is equivalent to saying that $L^2(\Omega, \mathfrak{B}, m)$ is separable as a metric space with $d(x, y)=\|x-y\|$ as its distance function. It is clear that the Lebesgue measure space is metrically separable.

A measure space $(\Omega', \mathfrak{B}', m')$ is an *extension* of another measure space $(\Omega, \mathfrak{B}, m)$ if $\Omega'=\Omega$, $\mathfrak{B}' \geqq \mathfrak{B}$ and $m'(B)=m(B)$ on $\mathfrak{B}$. The purpose of this paper is to prove, by constructing an example, the following

Proposition. There exists a metrically non-separable extension of the Lebesgue measure space whose character is $2^{\mathfrak{c}}$.

§ 2. We begin with some lemmas:

Lemma 1. Let S be an arbitrary set with $\mathfrak{p}(S)=\mathfrak{c}$[1]*. Then there exists a family $\mathfrak{S}=\{S_\gamma \mid \gamma \in \Gamma\}$ of subsets S_γ of S with the following properties:*

(1) $\mathfrak{p}(\mathfrak{S}) \equiv \mathfrak{p}(\Gamma)=2^{\mathfrak{c}}$,

1) $\mathfrak{p}(S)$ denotes the cardinal number of a set S,

(2) $\bigcap_{n=1}^{\infty} S_{\gamma_{2n-1}} \cap \bigcap_{n=1}^{\infty}(S-S_{\gamma_{2n}}) \neq \Theta$[1] *for any countable subset* $\Gamma_0 = \{\gamma_n \mid n=1, 2, \dots\}$ *of* Γ.

This Lemma is due to A. Tarski[2].

Lemma 2. There exists a family $\mathfrak{M}=\{M_\delta \mid \delta \in \Delta\}$ *of subsets* M_δ *of the interval* $\Omega=\{\omega \mid 0 \leqq \omega \leqq 1\}$ *of real numbers* ω *such that*

(3) $\mathfrak{p}(\mathfrak{M}) \equiv \mathfrak{p}(\Delta) = \mathfrak{c}$,

(4) $M_\gamma \cap M_\delta = \Theta, \quad \gamma \neq \delta$,

(5) $m^*(M_\delta)=1$ *for any* $\delta \in \Delta$, *where we denote by* $m^*(M)$ *the Lebesgue outer measure of a subset* M *of* Ω.

Proof. Let $\mathfrak{F}=\{F_\alpha \mid 0 \leqq \alpha < \omega_1\}$ be a well-ordering of all closed subsets F_α of Ω with $0 < m(F_\alpha) \leqq 1$, where ω_1 denotes the first ordinal number of the third class. Let us define a family $\mathfrak{N}=\{N_\alpha \mid 0 \leqq \alpha < \omega_1\}$ of null sets N_α with the following properties:

(6) $N_\alpha \subseteqq F_\alpha$ for any α,

(7) $N_\alpha \cap N_\beta = \Theta, \quad \alpha \neq \beta$,

(8) $\mathfrak{p}(N_\alpha)=\mathfrak{c}$ for any α.

In order to construct such a family by transfinite induction, let N_0 be an arbitrary subset of F_0 of measure zero with $\mathfrak{p}(N_0)=\mathfrak{c}$. Let now $0 < \alpha < \omega_1$, and assume that the family $\{N_\beta \mid 0 \leqq \beta < \alpha\}$ of null sets N_β is already defined. Since $\bigcup_{0 \leqq \beta < \alpha} N_\beta$ is a null set, there exists a null set N_α such that $N_\alpha \subseteqq F_\alpha - F_\alpha \cap \bigcup_{0 \leqq \beta < \alpha} N_\beta$ and $\mathfrak{p}(N_\alpha)=\mathfrak{c}$. It is clear that we can carry out the transfinite induction and thus obtain a family $\mathfrak{N}=\{N_\alpha \mid 0 \leqq \alpha < \omega_1\}$ with the required properties (6), (7) and (8). We notice that

(9) *for any measurable subset* B *of* Ω *with* $m(B) > 0$, *there exists an* α *such that* $N_\alpha \subseteqq B$.

Let further $N_\alpha = \{\omega_{\alpha\beta} \mid 0 \leqq \beta < \omega_1\}$ be a well-ordering of all elements of each N_α, where ω_1 is again the first ordinal number of the third class. If we put $M_\beta = \{\omega_{\alpha\beta} \mid 0 \leqq \alpha < \omega_1\}$, then the family $\mathfrak{M}=\{M_\beta \mid 0 \leqq \beta < \omega_1\}$ thus obtained is a required one. In fact, it is clear that the conditions (3) and (4) are satisfied. In order to show that $\mathfrak{M}$ has the property (5), assume that $m^*(M_\beta) < 1$ for some β. Then there would exist a measurable subset B of Ω with $m(B) > 0$ such that $M_\beta \cap B = \Theta$. This is, however, a contradiction since B contains some N_α and hence an element $\omega_{\alpha\beta} \in N_\alpha \cap M_\beta$. Thus $\mathfrak{M}$ must have the property (5), and this completes the proof of Lemma 2.

Lemma 3. There exists a family $\mathfrak{A}=\{A_\gamma \mid \gamma \in \Gamma\}$ *of subsets* A_γ *of the interval* $\Omega=\{\omega \mid 0 \leqq \omega \leqq 1\}$ *of real numbers* ω *with the following properties:*

(10) $\mathfrak{p}(\mathfrak{A}) \equiv \mathfrak{p}(\Gamma) = 2^{\mathfrak{c}}$,

1) Θ denotes the empty set.

2) A. Tarski, Fund. Math., **32** (1939).

(11) $m^*\left(\bigcap_{n=1}^{\infty} A_{\gamma_{2n-1}} \cap \bigcap_{n=1}^{\infty}(\Omega - A_{\gamma_{2n}})\right)=1$ *for any countable subset* $\Gamma_0 = \{\gamma_n \mid n=1, 2, \dots\}$ *of* Γ.

Lemma 3 is an immediate consequence of the combination of Lemmas 1 and 2.

§ 3. We are now in a position to construct a required example.

Let $\mathfrak{A}=\{A_\gamma \mid \gamma \in \Gamma\}$ be a family of subsets A_γ of the interval $\Omega=\{\omega \mid 0 \leqq \omega \leqq 1\}$ of real numbers ω with the properties (10) and (11) as obtained in Lemma 3. Let us then denote by $\mathfrak{E}=\{E\}$ the family of all subsets E of Ω of the form:

(12) $$E=\bigcup_{\{\varepsilon_1,\dots,\varepsilon_n\}} A_{\gamma_1}^{\varepsilon_1} \cap \cdots \cap A_{\gamma_n}^{\varepsilon_n} \cap B_{\varepsilon_1,\dots,\varepsilon_n},$$

where $\{\gamma_1, \cdots, \gamma_n\}$ is an arbitrary n-system from Γ (i.e. a finite subset of Γ consisting of n elements) (n is also an arbitrary positive integer), $\{B_{\varepsilon_1,\dots,\varepsilon_n} \mid \varepsilon_i=1 \text{ or } -1;\ i=1,\dots,n\}$ is an arbitrary 2^n-system from $\mathfrak{B}$ (=the family of all Lebesgue measurable subsets B of Ω), and A^ε means A or $\Omega-A$ according as $\varepsilon=1$ or -1. Further, $\bigcup_{\{\varepsilon_1,\dots,\varepsilon_n\}}$ denotes the union of 2^n sets which correspond to all possible combinations $\{\varepsilon_1, \cdots, \varepsilon_n\}$, $\varepsilon_i=1$ or -1; $i=1,\dots,n$, (n being fixed).

E is clearly a field which contains $\mathfrak{B}$, i.e. every measurable subset B of Ω is contained in $\mathfrak{E}$, and $E_1, E_2 \in \mathfrak{E}$ implies $E_1 \cup E_2$, $E_1 \cap E_2$, $\Omega - E_1 \in \mathfrak{E}$. Further, for any given $E \in \mathfrak{E}$ and an n-system $\{\gamma_1, \cdots, \gamma_n\} \subseteqq \Gamma$, the expression (12) is unique up to null sets in the following sense: if there exists another expression

(13) $$E=\bigcup_{\{\varepsilon_1,\dots,\varepsilon_n\}} A_{\gamma_1}^{\varepsilon_1} \cap \cdots \cap A_{\gamma_n}^{\varepsilon_n} \cap B'_{\varepsilon_1,\dots,\varepsilon_n}$$

with the same n-system $\{\gamma_1, \cdots, \gamma_n\} \subseteqq \Gamma$ but with possibly different $B'_{\varepsilon_1,\dots,\varepsilon_n}$, then $m(B_{\varepsilon_1,\dots,\varepsilon_n} \ominus B'_{\varepsilon_1,\dots,\varepsilon_n})=0$ for any $\{\varepsilon_1, \cdots, \varepsilon_n\}$. In fact, from (12) and (13) follows that

(14) $$A_{\gamma_1}^{\varepsilon_1} \cap \cdots \cap A_{\gamma_n}^{\varepsilon_n} \cap B_{\varepsilon_1,\dots,\varepsilon_n} = A_{\gamma_1}^{\varepsilon_1} \cap \cdots \cap A_{\gamma_n}^{\varepsilon_n} \cap B'_{\varepsilon_1,\dots,\varepsilon_n}$$

for any $\{\varepsilon_1, \cdots, \varepsilon_n\}$, which together with the relation $m^*(A_{\gamma_1}^{\varepsilon_1} \cap \cdots \cap A_{\gamma_n}^{\varepsilon_n}) = 1$ (which itself is a special case of (11)) imply that $m(B_{\varepsilon_1,\dots,\varepsilon_n} \ominus B'_{\varepsilon_1,\dots,\varepsilon_n}) = 0$ for any $\{\varepsilon_1, \cdots, \varepsilon_n\}$. In the same way it may be shown that if

(15) $$E=\bigcup_{\{\varepsilon_1,\dots,\varepsilon_n\}} A_{\gamma_1}^{\varepsilon_1} \cap \cdots \cap A_{\gamma_n}^{\varepsilon_n} \cap B_{\varepsilon_1,\dots,\varepsilon_n}$$
$$=\bigcup_{\{\varepsilon_1,\dots,\varepsilon_{n+p}\}} A_{\gamma_1}^{\varepsilon_1} \cap \cdots \cap A_{\gamma_{n+p}}^{\varepsilon_{n+p}} \cap B'_{\varepsilon_1,\dots,\varepsilon_{n+p}}$$

for some $(n+p)$-system $\{\gamma_1, \cdots, \gamma_{n+p}\} \subseteqq \Gamma$, 2^n-system $\{B_{\varepsilon_1,\dots,\varepsilon_n} \mid \varepsilon_i=1 \text{ or } -1;\ i=1,\dots,n\} \subseteqq \mathfrak{B}$ and 2^{n+p}-system $\{B'_{\varepsilon_1,\dots,\varepsilon_{n+p}} \mid \varepsilon_i=1 \text{ or } -1;\ i=1, \dots, n+p\} \subseteqq \mathfrak{B}$, then $m(B_{\varepsilon_1,\dots,\varepsilon_n} \ominus B'_{\varepsilon_1,\dots,\varepsilon_{n+p}})=0$ for any $\{\varepsilon_1, \cdots, \varepsilon_{n+p}\}$. Finally, if $E \in \mathfrak{E}$ is given by (12), $F \in \mathfrak{E}$ is given by

(16) $$F=\bigcup_{\{\varepsilon_1,\dots,\varepsilon_n\}} A_{\gamma_1}^{\varepsilon_1} \cap \cdots \cap A_{\gamma_n}^{\varepsilon_n} \cap B'_{\varepsilon_1,\dots,\varepsilon_n},$$

and if $E \cap F = \theta$, then $m(B_{\varepsilon_1,\dots,\varepsilon_n} \cap B'_{\varepsilon_1,\dots,\varepsilon_n})=0$ for any $\{\varepsilon_1, \cdots, \varepsilon_n\}$.

Let us now put

$$(17)\qquad \overline{m}(E)=\frac{1}{2^n}\sum\nolimits_{\{\varepsilon_1,\dots,\varepsilon_n\}} m(B_{\varepsilon_1,\dots,\varepsilon_n})$$

if $E\in\mathfrak{E}$ is given by (12), where $\sum_{\{\varepsilon_1,\dots,\varepsilon_n\}}$ denotes the sum of 2^n terms $m(B_{\varepsilon_1,\dots,\varepsilon_n})$ corresponding to all possible combinations $\{\varepsilon_1,\dots,\varepsilon_n\}$. It is then easy to see, by taking into considerations the facts observed above, that $\overline{m}(E)$ is uniquely defined for any $E\in\mathfrak{E}$ (although the expression (12) is not unique for any given $E\in\mathfrak{E}$), and further that $\overline{m}(E)$ is finitely additive on $\mathfrak{E}$.

We shall next show that $\overline{m}(E)$ can be extended to a countably additive measure $\overline{m}(\overline{B})$ defined on the Borel field $\overline{\mathfrak{B}}=\mathfrak{B}(\mathfrak{E})$ generated by $\mathfrak{E}$. For this purpose it suffices to show that

$$(18)\qquad E_k\in\mathfrak{E},\ k=1,2,\dots;\ E_1\geqq E_2\geqq\cdots;\ m(E_k)\geqq\delta>0,\ k=1,2,\dots \text{ imply } \bigcap_{k=1}^{\infty}E_k\neq\theta.$$

Without loss of generality, we may assume that there exists a countable set $\Gamma_0=\{\gamma_n\mid n=1,2,\dots\}\subseteqq\Gamma$, an increasing sequence $\{n_k\mid k=1,2,\dots\}$ of positive integers, and a sequence of 2^{n_k}-systems $\{B^{(k)}_{\varepsilon_1,\dots,\varepsilon_{n_k}}\mid \varepsilon_i=1$ or $-1;\ i=1,\dots,n_k\}$ such that

$$(19)\qquad E_k=\bigcup\nolimits_{\{\varepsilon_1,\dots,\varepsilon_{n_k}\}} A^{\varepsilon_1}_{\gamma_1}\cap\cdots\cap A^{\varepsilon_{n_k}}_{\gamma_{n_k}}\cap B^{(k)}_{\varepsilon_1,\dots,\varepsilon_{n_k}},$$

$$(20)\qquad B^{(k)}_{\varepsilon_1,\dots,\varepsilon_{n_k}}\geqq B^{(k+1)}_{\varepsilon_1,\dots,\varepsilon_{n_{k+1}}}$$

for any k and for any $\{\varepsilon_1,\dots,\varepsilon_{n_{k+1}}\}$. Since

$$(21)\qquad \overline{m}(E_k)=\frac{1}{2^{n_k}}\sum\nolimits_{\{\varepsilon_1,\dots,\varepsilon_{n_k}\}} m(B^{(k)}_{\varepsilon_1,\dots,\varepsilon_{n_k}})\geqq\delta>0$$

for each k, there exists, for each k, at least one combination $\{\varepsilon^{(k)}_1,\dots,\varepsilon^{(k)}_{n_k}\}$ such that

$$(22)\qquad m(B^{(k)}_{\varepsilon^{(k)}_1,\dots,\varepsilon^{(k)}_{n_k}})\geqq\delta>0\,.$$

It is then not difficult to see, by appealing to the relation (20), that there exists a sequence $\{\varepsilon^{(0)}_n\mid n=1,2,\dots\}$ $(\varepsilon^{(0)}_n=1$ or $-1,\ n=1,2,\dots)$ such that

$$(23)\qquad m(B^{(k)}_{\varepsilon^{(0)}_1,\dots,\varepsilon^{(0)}_{n_k}})\geqq\delta>0$$

for $k=1,2,\dots$. From this follows, again by appealing to (20), that

$$(24)\qquad m(\bigcap_{k=1}^{\infty}B^{(k)}_{\varepsilon^{(0)}_1,\dots,\varepsilon^{(0)}_{n_k}})\geqq\delta>0\,,$$

which together with the relation

$$(25)\qquad m^*(\bigcap_{n=1}^{\infty}A^{\varepsilon^{(0)}_n}_{\gamma_n})=1$$

(which itself is an immediate consequence of (11)) will imply

$$\text{(26)} \qquad \bigcap_{k=1}^{\infty} E_k \geqq \bigcap_{k=1}^{\infty} \left(A_{\gamma_1}^{\varepsilon_1^{(0)}} \cap \cdots \cap A_{\gamma_{n_k}}^{\varepsilon_{n_k}^{(0)}} \cap B_{\varepsilon_1^{(0)},\ldots,\varepsilon_{n_k}^{(0)}}^{(k)}\right)$$
$$= \bigcap_{n=1}^{\infty} A_{\gamma_n}^{\varepsilon_n^{(0)}} \cap \bigcap_{k=1}^{\infty} B_{\varepsilon_1^{(0)},\ldots,\varepsilon_{n_k}^{(0)}}^{(k)} \neq \theta\,.$$

Thus we see that $\overline{m}(E)$ can be extended to a countably additive measure $\overline{m}(\overline{B})$ defined on the Borel field $\overline{\mathfrak{B}} = \mathfrak{B}(\mathfrak{E})$ generated by $\mathfrak{E}$. It is easy to see that the measure space $(\Omega, \overline{\mathfrak{B}}, \overline{m})$ thus obtained has the character $2^{\mathfrak{c}}$. In fact, denoting by $\chi_\gamma(\omega)$ the characteristic function of the set A_γ and putting $\varphi_\gamma(\omega) = 2\chi_\gamma(\omega) - 1$ for any $\gamma \in \Gamma$, the relations $\overline{m}(A_\gamma) = \frac{1}{2}$, $\gamma \in \Gamma$, and $\overline{m}(A_\gamma \cap A_\delta) = \overline{m}\big(A_\gamma \cap (\Omega - A_\delta)\big) = \frac{1}{4}$, $\gamma \neq \delta$ (which themselves are the consequences of the definition (17) of $\overline{m}(E)$ on $\mathfrak{E}$), imply that $\{\varphi_\gamma(\omega) \mid \gamma \in \Gamma\}$ is an orthonormal system in $L^2(\Omega, \overline{\mathfrak{B}}, \overline{m})$. Thus the character of $(\Omega, \overline{\mathfrak{B}}, \overline{m})$ is $\geqq 2^{\mathfrak{c}}$. Since, on the other hand, $\overline{\mathfrak{B}}$ contains at most $2^{\mathfrak{c}}$ sets (in fact, there are only $2^{\mathfrak{c}}$ different subsets of Ω), we must have $\mathfrak{p}(\Omega, \overline{\mathfrak{B}}, \overline{m}) \leqq 2^{\mathfrak{c}}$. Since it is clear that $(\Omega, \overline{\mathfrak{B}}, \overline{m})$ is an extension of the Lebesgue measure space $(\Omega, \mathfrak{B}, m)$, so we finally see that $(\Omega, \overline{\mathfrak{B}}, \overline{m})$ is a required example.