43. On the Normal Stationary Process with no Hysteresis.

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§ 1. Let (Ω, P) be a probability field and $\mathfrak{M}$ a system of real valued random variables. $\mathfrak{M}$ is called to be *normal* or *of the Gaussian type*, if, for any $x_i(\omega)\in\mathfrak{M}$, $i=1,2,\dots,n$, the random variable $\big(x_1(\omega), x_2(\omega),\dots,x_n(\omega)\big)$ is subjected to an n-dimensional (sometimes perhaps degenerated) Gaussian distribution. *This condition is equivalent to the property that, for any* $x_i(\omega)\in\mathfrak{M}$, *and for any real* a_i, $\sum_{i=1}^{n} a_i x_i(\omega)$ *is normally distributed.* Let $x_i(\omega)$, $i=1,2,\dots,m$, $y_j(\omega)$, $j=1,2,\dots,n$, be elements in a normal system $\mathfrak{M}$. *Then the non-correlatedness of* $x_i(\omega)$ *and* $y_j(\omega)$ *for any,* i,j, $1\leqq i\leqq m$, $1\leqq j\leqq n$, *implies the independence of* $\big(x_1(\omega), x_2(\omega),\dots,x_m(\omega)\big)$ *and* $\big(y_1(\omega), y_2(\omega),\dots,y_n(\omega)\big)$.

Let $x(t,\omega)$, $-\infty<t<\infty$, be a stochastic process. If the system of $x(t,\omega)$, $-\infty<t<\infty$, is normal, then the process will be said to be *normal*. If the (conditional) probability law of $x(t,\omega)$ under the condition that $x(t_1,\omega), x(t_2,\omega),\dots,x(t_n,\omega)$ should be given depends only on the value $x(t_n,\omega)$ for any $t_1<t_2<\dots<t_n<t$, we say that $x(t,\omega)$ *has no hysteresis* or *is a simple Markoff process.* This terminology is applied to the case of a stochastic sequence $x(k,\omega)$, $k=0,\pm1,\pm2,\dots$.

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§ 2. *The form of the correlation function.*

Theorem 1. Let $x(k,\omega)$ *be a normal stationary (in the sense of A. Khintchine) sequence. A necessary and sufficient condition that* $x(k,\omega)$ *should have no hysteresis is that its correlation function* $\rho(k)$ *is of the form* α^k, $-1\leqq\alpha\leqq1$.

Proof. In order to avoid trivial complications we assume $E_\omega\big(x(k,\omega)\big)=0$, and $E_\omega\big(x(k,\omega)^2\big)=1$. If we define $\big(f(\omega), g(\omega)\big)$ by $E_\omega\big(f(\omega)g(\omega)\big)$, the closed linear subspace determined by the set $x(k,\omega)$, $k=0,\pm1,\pm2,\dots$, is considered as a Hilbert space, where *orthogonality implies (stochastic) independence* (Cf. § 1).

Sufficiency. For the proof it is sufficient to show that the conditional probability law of $x(k,\omega)$ under the condition that $x(k-i,\omega)=\xi_i$, $i=1,2,\dots,n$, depends only on ξ_1.

We put $y(k,\omega)=x(k,\omega)-\alpha x(k-1,\omega)$. Then we have $E_\omega\big(y(k,\omega)x(k-i,\omega)\big)=\alpha^i-\alpha\alpha^{i-1}=0$, $i=1,2,\dots$. Since the sequence is normal, $y(k,\omega)$ is independent of $\big(x(k-1,\omega), x(k-2,\omega),\dots,x(k-n,\omega)\big)$. Therefore the probability law of $y(k,\omega)$ is invariant even if we add the condition: $x(k-i,\omega)=\xi_i$, $i=1,2,\dots,n$. Therefore the probability law of $x(k,\omega)$ i.e. of $\alpha x(k-1,\omega)+y(k,\omega)$ under the same condition depends only on ξ_1.

Necessity. We fix a natural number n arbitrarily. Let L denote the linear manifold determined by the set $x(-i,\omega)$, $i=1,2,\dots,n$ and $\sum_{i=1}^{n} a_i x(-i,\omega)$ the orthogonal projection of $x(o,\omega)$ into L. Now we put

$$x(o,\omega)=\sum_{i=1}^{n} a_i x(-i,\omega)+y(\omega) \tag{1}$$

$y(\omega)$, being orthogonal to L, is independent of $\big(x(-1,\omega), x(-2,\omega),\dots, x(-n,\omega)\big)$. We consider two conditions (A) $x(-i,\omega)=a_i$, $i=1,2,\dots,n$, (B) $x(-1,\omega)=a_1$, $x(-i,\omega)=0$, $i=2,3,\dots,n$. The expectation of $x(0\,\omega)$ is equal to $\sum_{i=1}^{n} a_i^2$ under (A), while it is a_1^2 under (B). Since the sequence has no hysteresis the two values must be coincident, from which follows that $a_i=0$, $i=2,3,\dots,n$. Thus the identity (1) becomes $x(o,\omega)=a_1x(-1,\omega)+y(\omega)$. Therefore we have

$$\begin{aligned}\rho(1)&=E_\omega\big(x(o,\omega)x(-1,\omega)\big)\\ &=a_1E_\omega\big(x(-1,\omega)^2\big)+E_\omega\big(y(\omega)x(-1,\omega)\big)=a_1\end{aligned} \tag{2}$$

Consequently we can see that $x(o,\omega)-\rho(1)x(-1,\omega)$, being equal to $y(\omega)$, is independent of $x(-n,\omega)$, i.e. that $E_\omega\big(x(-n,\omega)\big(x(o,\omega)-\rho(1)x(-1,\omega)\big)\big)=0$, and so that $\rho(n)-\rho(1)\rho(n-1)=0$. Since $\rho(0)=1$, we have $\rho(n)=\alpha^n$, $n=0,1,2,\dots$, where $\alpha=\rho(1)$, and so $-1\leqq\alpha\leqq 1$. $\rho(n)$, as an even function, is equal to $\alpha^{|n|}$ for any integer n.

Theorem 2. Let $x(t,\omega)$ be any normal stationary continuous $\big($in the strong topology in $L^2(\Omega,P)\big)$ process. The necessary and sufficient condition that it should have no hysteresis is that its correlation function $\rho(\tau)$ is of the form $e^{-\alpha|\tau|}$, $\alpha\geqq 0$.

Proof. We prove only the necessity, since the sufficiency can be shown in the same manner as before. By Theorem 1 we have clearly $\rho(\tau)=\rho\left(\frac{\tau}{2}\right)^2>0$ and $\rho\left(\frac{m}{n}\right)=\rho(1)^{\left|\frac{m}{n}\right|}=e^{-\alpha\left|\frac{m}{n}\right|}$ $\big(\alpha=-\log\rho(1)\geqq 0\big)$. By the continuity of the process we obtain that of $\rho(\tau)$ and so $\rho(\tau)=e^{-\alpha|\tau|}$.

§ 3. *The form of the sequence (or process).*

Theorem 3. Any normal stationary sequence with no hysteresis is expressible in one of the following three forms and its converse is also true.

$$x(k,\omega)=x(\omega)\,,\qquad k=0,\pm1,\pm2,\dots \tag{3}$$

$$x(k,\omega)=(-1)^k x(\omega)\,,\qquad k=0,\pm1,\pm2,\dots \tag{4}$$

In these cases $x(\omega)$ denotes a normally distributed random variable.

$$x(k,\omega)=\sum_{n=-\infty}^{k}\alpha^{k-n}y(n,\omega)\,,\qquad k=0,\pm1,\pm2,\dots, \tag{5}$$

where $-1<\alpha<1$ and $y(k,\omega)$, $k=0,\pm1,\pm2,$ is an independent *sequence of random variables subjected to the same Gaussian distribution and is expressible by $x(k, \omega)$ as follows:*

$$(5') \qquad y(k, \omega)=x(k, \omega)-\alpha x(k-1, \omega).$$

Proof. We may assume $E_\omega\big(x(k, \omega)\big)=0$ and $E_\omega\big(x(k, \omega)^2\big)=1$ with no loss of generality. By Theorem 1 the correlation function $\rho(k)$ is equal to α^k, $-1 \leqq \alpha \leqq 1$. In case $\alpha=1$, $x(k, \omega)$ is of the form (3), since $E_\omega\Big(\big(x(k, \omega)-x(0, \omega)\big)^2\Big)=2-2\rho(k)=0$. Similarly we obtain (4) in case $\alpha=-1$.

Suppose that $-1<\alpha<1$. We define $y(k, \omega)$ by (5′). The probability law of $y(k, \omega)$ is a Gaussian distribution. But we have, for $k>h$,

$$\begin{aligned} E_\omega\big(y(k, \omega)y(h, \omega)\big) &= \rho(k-h)-\alpha\rho(k-h-1)-\alpha\rho(k-h+1)+\alpha^2\rho(k-h) \\ &= \alpha^{k-h}-\alpha\cdot\alpha^{k-h-1}-\alpha\cdot\alpha^{k-h+1}+\alpha^2\alpha^{k-h} \\ &= 0. \end{aligned}$$

Therefore $y(k, \omega)$, $k=0, \pm 1, \pm 2, \ldots$, is an independent sequence. From (5′) we deduce, by the principle of recursion,

$$\begin{aligned} x(k, \omega) &= y(k, \omega)+\alpha x(k-1, \omega) \\ &= \sum_{n=-N}^{k} \alpha^{k-n} y(n, \omega)+\alpha^{k+N+1}x(-N-1, \omega). \end{aligned}$$

But $E_\omega\Big(\big(\alpha^{k+N+1}x(-N-1, \omega)\big)^2\Big)=\alpha^{2(k+N+1)}$ tends to 0, as $N\to\infty$. Thus we obtain (5).

The converse proposition is evident.

Theorem 4. Any continuous stationary process with no hysteresis is expressible in one of the following two forms, and its converse is also true.

$$(6) \qquad x(t, \omega)=x(\omega), \qquad -\infty<t<\infty,$$

where $x(\omega)$ is a normally distributed random variable.

$$(7) \qquad x(t, \omega)=\int_{-\infty}^{t} e^{-\alpha(t-\tau)}d_\tau y(\tau, \omega),$$

where $y(t, \omega)$ is a brownian motion, i.e. a temporally and spatially homogeneous process with no moving discontinuity, and is expressible by $x(t, \omega)$ as follows:

$$(7') \qquad y(t, \omega)-y(s, \omega)=x(t, \omega)-x(s, \omega)+\alpha\int_{s}^{t} x(\tau, \omega)d\tau,$$

and the integral in (7) is to be understood as a Riemann-Stieltjes integral in the sense of the strong topology in $L^2(\Omega, P)$.

Proof. We may assume $E_\omega\big(x(t, \omega)\big)=0$ and $E_\omega\big(x(t, \omega)^2\big)=1$ with no loss of generality. By Theorem 1 the correlation function is equal to $e^{-\alpha|\tau|}$, $\alpha \geqq 0$. If $\alpha=0$, then the process is clearly expressible in (6).

Suppose that $\alpha>0$. We define $y(t, \omega)$ by (7′). Making use of the identity $E_\omega\big(x(t,\omega)x(s,\omega)\big)=\rho(t-s)=e^{-\alpha|t-s|}$, we can prove, by easy calculations,

$$E_\omega\big(y(t,\omega)-y(s,\omega)\big)=0 \qquad (s<t), \tag{8}$$

$$E_\omega\Big(\big(y(t,\omega)-y(s,\omega)\big)^2\Big)=2\alpha(t-s) \qquad (s<t), \qquad \text{and} \tag{9}$$

$$E_\omega\Big(\big(y(t,\omega)-y(s,\omega)\big)\big(y(s,\omega)-y(u,\omega)\big)\Big)=0 \qquad (u<s<t). \tag{10}$$

Since the process $x(t,\omega)$ is normal, (10) implies that $y(t,\omega)-y(s,\omega)$ and $y(s,\omega)-y(u,\omega)$ are independent and normally distributed. Therefore the process $y(t,\omega)$ is a brownian motion.

The right side of (7) is clearly equal to

$$\begin{aligned}&\int_{-\infty}^{t} e^{-\alpha(t-\tau)}d_\tau x(\tau,\omega)+\alpha\int_{-\infty}^{t} e^{-\alpha(t-\tau)}x(\tau,\omega)d\tau\\ &\quad=\big[e^{-\alpha(t-\tau)}x(\tau,\omega)\big]_{-\infty}^{t}-\alpha\int_{-\infty}^{t} e^{-\alpha(t-\tau)}x(\tau,\omega)d\tau+\alpha\int_{-\infty}^{t} e^{-\alpha(t-\tau)}x(\tau,\omega)d\tau\\ &\quad=x(t,\omega).\end{aligned}$$

Thus we obtain (7).

The converse proposition is evident*.

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