# 73. A Generalized Limit.

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We are concerned with the definition and existence of a generalized limit, which possesses all the properties of the Banach limit in much generalized form except the invariance under translations of sequences. The invariance is closely related to the nature of the arithmetic mean.

Let $\Delta$ be a directed system, *i. e.* a partially ordered non-empty set, for any pair of whose elements $\alpha, \beta$ there exists a third element $\gamma$ satisfying $\alpha \leqq \gamma$ and $\beta \leqq \gamma$. We shall say that a condition $C(\alpha)$ concerning a variable $\alpha \in \Delta$, is *finally* satisfied or that $C(\alpha)$ is satisfied *for large* $\alpha$, when and only when there exists an $\alpha_0 \in \Delta$ such that $C(\alpha)$ is satisfied for all $\alpha \geqq \alpha_0$. If each of a finite number of conditions $C_1(\alpha), \dots, C_n(\alpha)$ is finally satisfied, their conjunction $C_1(\alpha)\&\dots C_n(\alpha)$ is also finally satisfied.

A *sequence* is a function $\varphi(\alpha)$ defined for large $\alpha$ in $\Delta$. Let $E$ be a Hausdorff space and let $\varphi(\alpha)$ be a sequence of its points: $\varphi(\delta) \in E$. A point $p \in E$ will be called the limit of the sequence, if and only if every neighbourhood of $p$ finally contains $\varphi(\alpha)$. In such a case the set $\Phi(\alpha) = \overline{\{\varphi(\beta) \mid \beta \geqq \alpha\}}$ (bar indicates the closure in $E$) is finally bicompact, if $E$ is locally bicompact. $E$ will be supposed locally bicompact in what follows.

Let $S$ be the totality of sequences $\varphi(\alpha)$ of points in $E$ such that $\Phi(\alpha)$ are finally bicompact, and let $S' \leqq S$. A *generalized limit* of sequences $\varphi \in S'$ is a function $L(\varphi)$ defined for all $\varphi \in S'$, with values in $E$ and satisfying the following conditions:

1) If $\varphi \in S'$ and $\Phi(\alpha) = \overline{\{\varphi(\beta) \mid \beta \geqq \alpha\}}$, then $L(\varphi) \in \Phi(\alpha)$.

2) If $\varphi, \psi \in S'$ and $\varphi(\alpha) = \psi(\alpha)$ for large $\alpha$, then $L(\varphi) = L(\psi)$.

3) Let $\psi, \varphi_1, \dots, \varphi_n \in S'$ and let $E^n$ be the cartesian product $E \times \dots \times E$ with the usual topology. If a function $f$ defined in $E^n$, with values in $E$, is continuous at $\big(L(\varphi_1), \dots, L(\varphi_n)\big)$, and if $\psi \in S'$ where $\psi(\alpha) = f\big(\varphi_1(\alpha), \dots, \varphi_n(\alpha)\big)$ for large $\alpha$, then $L(\psi) = f\big(L(\varphi_1), \dots, L(\varphi_n)\big)$.

It is easily seen (cf. 1) that $L(\varphi)$ coincides with the ordinary limit of $\varphi$, if the latter exists.

For the proof of existence of $L$, let us define $\varphi \equiv \psi$ when and only when $\varphi(\alpha) = \psi(\alpha)$ for large $\alpha$, and let us first consider the case when for any distinct $\varphi, \psi \in S'$ $\varphi \equiv \psi$ does not hold. Let us further assume that for every $\varphi \in S'$ $\varphi(\alpha)$ is defined for all $\alpha \in \Delta$ and that $\overline{\varphi(\Delta)}$ is bicompact. The Hausdorff spaces $\overline{\varphi(\Delta)}$ being bicompact, their cartesian product $P = \prod_{\varphi \in S} \overline{\varphi(\Delta)}$ with weak topology is a bicompact Hausdorff space, which consists by definition, of all the functions $F(\varphi)$ defined for all

$\varphi \in S'$, with values in $\overline{\varphi(\Delta)}$; it contains all the functions $F_\alpha(\varphi)=\varphi(\alpha)$, $\alpha \in \Delta$. Put $P(\alpha)=\{F_\beta \mid \beta \geqq \alpha\}$. Then the intersection of a finite number of sets $P(\alpha_1), \dots, P(\alpha_n)$ is non-empty, for, $\Delta$ being a directed system, there exists an element $\beta \in \Delta$ satisfying $\beta \geqq \alpha_1, \dots, \beta \geqq \alpha_n$ simultaneously. It follows that the intersection $D=\bigcap_{\alpha \in \Delta} \overline{P(\alpha)}$ of closed subsets of $P$ is not empty. Let $L$ be an element of $D$. It is sufficient to show 1) and 3), since $\varphi \equiv \psi$ never holds in $S'$. *Proof of* 1): Every neighbourhood of $L$ contains an element of $P(\alpha)$, and, in particular, every neighbourhood of $L(\varphi)$ contains an element of $\{F_\beta(\varphi) \mid F_\beta \in P(\alpha)\}=\{\varphi(\beta) \mid \beta \geqq \alpha\}$. Q. E. D. *Proof of* 3): Since $E$ is a Hausdorff space, it is sufficient to show that every neighbourhood $V$ of $L(\psi)$ intersects with every neighbourhood $U$ of $f\big(L(\varphi_1), \dots, L(\varphi_n)\big)$. Let $V$ and $U$ be given. Then, $f$ being continuous at the point $\big(L(\varphi_1), \dots, L(\varphi_n)\big)$, there exist a system of neighbourhoods $V_1, \dots, V_n$ of $L(\varphi_1), \dots, L(\varphi_n)$ respectively such that $f(V_1 \times \cdots \times V_n) \leqq U$. The neighbourhoods $V, V_1, \dots, V_n$ determines a neighbourhood of the point $L$ in $P$, and there exists in it an element of $P(\alpha_0)$, where $\alpha_0$ is an element of $\Delta$ such that $\psi(\alpha)=f\big(\varphi_1(\alpha), \dots, \varphi_n(\alpha)\big)$ holds for all $\alpha \geqq \alpha_0$. Let $F_\alpha$ be such an element in the neighbourhood. Then $F_\alpha(\psi)=\psi(\alpha)=f\big(\varphi_1(\alpha), \dots, \varphi_n(\alpha)\big)=f\big(F_\alpha(\varphi_1), \dots, F_\alpha(\varphi_n)\big) \in U$, $F_\alpha(\psi) \in V$, and so $U \frown V \ni F_\alpha$, *i.e.* $U$ and $V$ intersect. Q. E. D.

Now let us go over to the general case, and let us define a generalized limit $L$ on $S$. The relation $\varphi \equiv \psi$ is obviously an equivalence relation, and it induces a classification of sequences in $S$. Every class contains, as is easily seen, a sequence $\varphi(\alpha)$, defined for all $\alpha \in \Delta$, such that $\varphi(\Delta)$ is bicompact. Select one and only one such representative sequence from each class, and collect them in an aggregate $S'$. Then $S'$ is such a set that has been considered, and there exists a generalized limit $L'$ on it. We have now only to put $L(\varphi)=L'(\varphi')$ when $\varphi \equiv \varphi'$, $\varphi \in S$, $\varphi' \in S'$; proof of 1), 2), 3) are easy, and so omitted.