

116. On the Diophantine Analysis of Algebraic Functions.

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Tue's and Siegel's theorems are very important and significant in the recent development of the diophantine analysis. We see that the greater part of these theories will hold good even if algebraic functions are considered, instead of algebraic numbers. The purpose of this paper is to treat the outline of this analogy. Details will be discussed in another paper.

We consider an algebraic function $\omega(t)$ with elements of a corpus \mathcal{Q} as coefficients. The Laurent's expansion of $\omega(t)$ at infinity is of the form

$$\omega(t) = \sum_{n=p}^{-\infty} a_n t^n = a_p t^p + a_{p-1} t^{p-1} \dots + a_0 + \frac{a_{-1}}{t} + \frac{a_{-2}}{t^2} + \dots .$$

$$a_p \neq 0 .$$

Now we take a fixed real number $e > 1$ and put

$$\varphi(\omega(t)) = e^p \quad \text{for} \quad \omega(t) \neq 0, \quad \varphi(0) = 0 .$$

Then φ gives an evaluation (Bewertung) of the corpus of algebraic functions $\omega(t)$, such that

$$(1) \quad \varphi(\omega_1(t), \omega_2(t)) = \varphi(\omega_1(t)) \cdot \varphi(\omega_2(t)) ,$$

$$(2) \quad \varphi(\omega_1(t) + \omega_2(t)) \leq \max(\varphi(\omega_1(t)), \varphi(\omega_2(t))) .$$

In (2) the inequality happens eventually in the case $\varphi(\omega_1(t)) = \varphi(\omega_2(t))$. If we denote the integral part $a_p t^p + a_{p-1} t^{p-1} + \dots + a_0$ of $\omega(t)$ with the notation $[\omega(t)]$, we can define the continued fraction $[A_0(t), A_1(t), A_2(t), \dots]$ of an algebraic function $\omega(t)$, putting

$$A_0(t) = [\omega(t)] , \quad \omega(t) = A_0(t) + \frac{1}{\omega_1(t)} ,$$

$$[\omega_1(t)] = A_1(t) , \quad \omega_1(t) = A_1(t) + \frac{1}{\omega_2(t)} ,$$

$$[\omega_2(t)] = A_2(t) , \dots .$$

Then the fundamental theory of continued fraction can be applied without modification.

Theorem 1. (Lagrange). The necessary and sufficient condition that the continued fraction of $\omega(t)$ be recurrent is as follows :

1° $\omega(t)$ satisfies an irreducible quadratic equation with integral coefficients

$$a(t)\omega^2 + b(t)\omega + c(t) = 0 ;$$

2° The diophantine equation $x^2 - d(t)y^2 = 1$, $d(t) = b^2(t) - 4a(t)c(t)$ has at least one integral solution $x(t)$, $y(t)$ besides the trivial solutions $x = \pm 1$, $y = 0$.

Theorem 2. (Lagrange). If a diophantine equation $a(t)x^2 + b(t)xy + c(t)y^2 = k(t)$ has at least one integral solution, it has infinitely many solutions, when and only when the corresponding diophantine equation $x^2 - d(t)y^2 = 1$, $d(t) = b^2(t) - 4a(t)c(t)$ has non-trivial solutions.

We must remark that a Fermat's equation $x^2 - d(t)y^2 = 1$ does not always have non-trivial integral solutions, which is not the case in Fermat's equations of numbers. This discordance comes from the following fact: The number of integers, whose absolute values are less than a positive constant M , is finite, while that of polynomials which satisfy $\varphi(p(t)) < M$ is not always finite. Therefore, this discordance vanishes when \mathcal{Q} is a finite corpus, namely a Galois field.

If an algebraic function η satisfies the irreducible equation $a_0(t)\eta^h + a_1(t)\eta^{h-1} + \dots + a_n(t) = 0$, we call the number $H(\eta) = \max_{i=0,1,2,\dots,h} \varphi(a_i(t))$ the height of η . Now we introduce the several notations:

K : a fixed corpus of algebraic functions;

η : a primitive function of K ;

ξ : an algebraic function of the n -th degree, which satisfies an irreducible equation of the d -th degree in K .

With the help of these notations, we can state the principal theorems which are analogous to Siegel's¹⁾.

Theorem 3. (Siegel). s be a positive integer $< d$, and θ be any positive number. Then we can choose a positive number M , which depend on ξ, θ and K , so that the inequality

$$\varphi(\xi - \eta) > \frac{1}{H(\eta)^{\frac{d}{s+1} + s + \theta}}$$

holds for any primitive function η of the height $H(\eta) > M$.

Theorem 4. (Siegel). s be a positive integer $< n$. Then we can choose a positive number M , which depends on ξ, h and θ , so that the inequality

$$\varphi(\xi - \eta) > \frac{1}{H(\eta)^{h\left(\frac{n}{s+1} + s\right)\theta}}$$

holds for any algebraic function of the h -th degree, whose height $H(\eta)$ is greater than M .

The next lemma 1 plays the most important part in the development of the theory.

Lemma 1. Let P be a corpus of algebraic functions, ξ an algebraic integral function, which is of the d -th degree with respect to P and s a positive integer $< d$ and ϑ any positive number. Further, let m denote the integral part of $\left(\frac{d+\vartheta}{s+1} - 1\right)r$, where r is any posi-

1) Siegel, C. L. Approximation algebraischer Zahlen. Math. Z. **10** (1921), 173-213.

tive integer. Then, if ξ and P have no branch point at infinity, there exists a polynomial $R(x, y)$, which satisfies the following conditions:

(1) $R(x, y)$ has the form $\sum_{\alpha=0}^{m+r} \sum_{\beta=0}^s b_{\alpha, \beta}(t) x^\alpha y^\beta$, where $b_{\alpha, \beta}(t)$ are functions of P .

(2) If we put $R_\lambda(x, y) = \frac{1}{\lambda!} \frac{\partial^\lambda R(x, y)}{\partial x^\lambda}$, then $R_\lambda(\xi, \xi) = 0$ for $\lambda = 0, 1, 2, \dots, r-1$.

Therefore, $R(x, \xi)$ can be put in the form $R(x, \xi) = (x - \xi)^r s(x)$, where $s(x)$ is a polynomial of x .

(3) We can choose a positive number c properly, which depends on ξ, P, ϑ and is independent of r, s , so that the inequality $\varphi(b_{\alpha, \beta}(t)) < c^r$ holds for any $b_{\alpha, \beta}(t)$.

Siegel proved this lemma 1 with the help of the so-called "Dirichlet's principle" (Dirichletsches Schubfachschlussz). Unfortunately this principle is helpless in our case; we use the following lemma 2 instead of Dirichlet's principle.

S be a set of polynomials $Q(x, y)$ of x, y in P , which is a linear modul of finite rank with respect to the corpus of numbers. S_k be the subset of S , which consists of all elements $Q(x, y)$ of S with the condition $\varphi(Q(\xi, \xi)) \leq e^k$. Then S_k is a linear modul of finite rank m_k .

Lemma 2. Between the ranks of S_k and S_{k-1} , there exists the relation $m_k = m_{k-1} + 1$ or $m_k = m_{k-1}$.

By lemma 2, the proof of the lemma 1 runs quite parallel as in the Siegel's paper.

We assume in the lemma 1 that ξ and P have no branch point at infinity. In order to remove this restriction, it is sufficient to put $t = u^q$, choosing a positive integer q properly. The point at infinity is no more a branch point of ξ or P , if we consider u as the independent variable instead of t .

From Siegel's theorem, we can deduce the extremely interesting theorem which corresponds to Thue's theorem. The method of this deduction is quite analogous to the case of algebraic numbers.

Theorem 5. (Thue). $A(x, y) = a_0(t)x^n + a_1(t)x^{n-1}y + \dots + a_n(t)y^n$ be an irreducible homogeneous form of x, y with polynomials $a_0(t), a_1(t), \dots, a_n(t)$ as coefficients. If $n \geq 3$, any solution $x(t), y(t)$ of the indeterminate equation $A(x, y) = d(t)$, where $d(t)$ is a polynomial of t , must satisfy the condition $\varphi(x(t)) < M, \varphi(y(t)) < M$, for a sufficiently large number M . In other words, if we choose a positive number M sufficiently large, the indeterminate equation has no polynomial solution $x(t), y(t)$, such as $\varphi(x(t)) > M, \varphi(y(t)) > M$.

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