109. *Stochastic Integral.**

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1. *Introduction.* Let (Ω, P) be any probability field, and $g(t, \omega)$, $0 \leqq t \leqq 1$, $\omega \in \Omega$, be any *brownian motion*[1] on (Ω, P) i.e. a (real) stochastic differential process with no moving discontinuity such that $\mathscr{E}\big(g(s, \omega)-g(t, \omega)\big)=0$[2] and $\mathscr{E}\big(g(s, \omega)-g(t, \omega)\big)^2=|s-t|$. In this note we shall investigate an integral $\int_0^t f(\tau, \omega)\, d_\tau g(\tau, \omega)$ for any element $f(t, \omega)$ in a functional class S^* which will be defined in § 2; the particular case in which $f(t, \omega)$ does not depend upon ω has already been treated by Paley and Wiener[3].

In § 2 we shall give the definition and prove fundamental properties concerning this integral. In § 3 we shall establish three theorems which give sufficient conditions for integrability. In § 4 we give an example, which will show a somewhat singular property of our integral.

2. *Definition and Properties.* For brevity we define the classes of measurable functions defined on $[0, 1] \times \Omega$: G, $S(t_0, t_1, \dots, t_n)$, S and S^* respectively as the classes of $f(t, \omega)$ satisfying the corresponding conditions, as follows,

G: $f(\tau, \omega)$, $g(\tau, \omega)$, $0 \leqq \tau \leqq t$, are independent of $g(\sigma, \omega)-g(t, \omega)$, $t \leqq \sigma \leqq 1$, for any t, $g(\tau, \omega)$ being the above mentioned brownian motion,

$S(t_0, t_1, \dots, t_n)$, $0=t_0<t_1<\dots<t_n=1$: $f(t, \omega) \in G \wedge L_2([0, 1] \times \Omega)$ and $f(t, \omega)=f(t_{i-1}, \omega)$, $t_{i-1} \leqq t < t_i$, $i=1, 2, \dots, n$,

S: $f(t, \omega)$ belongs to $S(t_0, \dots, t_n)$ for a system $t_0, t_1, \dots, t_n$ which may depend upon $f(t, \omega)$; in other words $S \equiv \cup S(t_0, t_1, \dots, t_n)$,

S^*: $f(t, \omega) \in G$ and for any ε there exists $h(t, \omega) \in \bar{S}$[4] such that

$$P\{\omega; f(t, \omega)=h(t, \omega) \text{ for any } t\} > 1-\varepsilon.$$

At first for $f(t, \omega) \in S$ we define the stochastic integral $\int_0^t f(\tau, \omega)\, d_\tau g(\tau, \omega)$ (for brevity denote it by $I(t, \omega; f)$) as follows:

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1) C. P. Lévy: Théorie de l'addition des variable aléatoire, P. 167, 1937, and also J. L. Doob: Stochastic processes depending on a continuous parameter, Trans., Amer. Math. Soc. vol. 42, Theorem 3.9.

2) $\mathscr{E}$ denotes the mathematical expectation, viz. $\mathscr{E} f(\omega)=\int_\Omega f(\omega)P(d\omega)$.

3) R. E. A. G. Paley and N. Wiener, Fourier transforms in the complex domain, Amer. Math. Soc. Coll. Publ. (1934), Chap. IX.

4) $\bar{S}$ means the closure of S with respect to the norm in $L_2([0, 1] \times \Omega)$.

$$(2.1)\qquad I(t,\omega;f)=\sum_{i=1}^{k}f(t_{i-1},\omega)\Big(g(t_i,\omega)-g(t_{i-1},\omega)\Big)+f(t_k,\omega)\Big(g(t,\omega)-g(t_k,\omega)\Big)$$

for $t_k \leqq t \leqq t_{k+1}$, if $f(t,\omega)\in S(t_0, t_1, \dots, t_n)$; this definition is independent of the special choice of $S(t_0, t_1, \dots, t_n)$. We have the

Theorem 2.1.

(*L*) $I(t,\omega;af+bg)=aI(t,\omega;f)+bI(t,\omega;g)$

(*N*) $I(t,\omega;1)=g(t,\omega)-g(0,\omega)$,

(*C*) *$I(t,\omega;f)$ is a continuous function of t with P-measure 1,*

(*I*) $\|I(t,\omega;f)\|_{\Omega}^{2}$[1] $=\|f(\tau,\omega)\|^2_{[0,t]\times\Omega}$ *for any* t, $0\leqq t\leqq 1$,

(*B*) $P\{\omega;\sup_{0\leqq t\leqq 1}|I(t,\omega;f)|\geqq b\}\leqq \frac{1}{b^2}\|f(\tau,\omega)\|^2_{[0,1]\times\Omega}$,

and

(*J*) *if $f(t,\omega)=h(t,\omega)$, $0\leqq t\leqq 1$ for any $\omega\in\Omega_1$, Ω_1 being any P-measurable subset of Ω, then $I(t,\omega;f)=I(t,\omega;h)$, $0\leqq t\leqq 1$, almost everywhere in Ω_1.*

Proof. (*L*), (*N*), (*C*) and (*J*) are evident by the definition. In order to show (*I*) we may assume $t=t_k$ with no loss of generality. The left side of (*I*) is the expectation of $I(t,\omega,f)^2$, say $\mathscr{E}\, I(t,\omega;f)^2$.

$$(2.2)\qquad \mathscr{E}\, I(t,\omega;1)^2=\sum_{i=1}^{k}\mathscr{E} f(t_{i-1},\omega)^2\Big(g(t_i,\omega)-g(t_{i-1},\omega)\Big)^2+2\sum_{i<j\leqq k}\mathscr{E} f(t_{i-1},\omega)f(t_{j-1},\omega)\Big(g(t_i,\omega)-g(t_{i-1},\omega)\Big)\Big(g(t_j,\omega)-g(t_{j-1},\omega)\Big).$$

In order to calculate this right side we shall achieve preliminary calculations. For brevity write f_t and g_t for $f(t,\omega)$ and $g(t,\omega)$ respectively. Since $f(t,\omega)\in G\frown L_2([0,1]\times\Omega)$, we have, for $t<s<u<v$,

$$\mathscr{E}\,|f_tf_u(g_s-g_t)|\leqq\sqrt{\mathscr{E} f_u^2\,\mathscr{E} f_t^2(g_s-g_t)^2}=\sqrt{\mathscr{E} f_u^2\,\mathscr{E} f_t^2\,\mathscr{E}(g_s-g_t)^2}<\infty,$$

$$\mathscr{E}\,|f_tf_u(g_s-g_t)(g_v-g_u)|=\mathscr{E}\,|f_tf_u(g_s-g_t)|\,\mathscr{E}\,|g_v-g_u|<\infty,$$

$$\mathscr{E} f_tf_u(g_s-g_t)(g_v-g_u)=\mathscr{E} f_tf_u(g_s-g_t)\,\mathscr{E}(g_v-g_u)=0.$$

Therefore we obtain

$$\mathscr{E}\, I(t,\omega;f)^2=\sum_{i=1}^{k}\mathscr{E} f(t_{i-1},\omega)^2(t_i-t_{i-1})=\int_0^{t_k}\int_{\Omega}f(t,\omega)^2P(d\omega)d\tau,$$

i.e. $\|I(t,\omega;f)\|^2_{\Omega}=\|f(t,\omega)\|^2_{[0,t]\times\Omega}$

For the proof of (*B*) we state the

Lemma 2.1. Let $y_i(\omega)$, $x_i(\omega)$, $i=1,2,\dots\dots,n$, be any random variables. We assume, for $i=1,2,\dots,n$, that $y_1(\omega)$, $x_1(\omega)$, $\dots$, $y_{i-1}(\omega)$, $x_{i-1}(\omega)$, $y_i(\omega)$ are independent of $x_i(\omega)$, $x_{i+1}(\omega)$, $\dots$, $x_n(\omega)$. Then we have

$$(2.3)\qquad P\{\omega;\max_{1\leqq i\leqq n}|y_1(\omega)x_1(\omega)+\cdots+y_i(\omega)x_i(\omega)|\geqq b\}\leqq\frac{1}{b^2}\mathscr{E}\Big(y_1(\omega)x_1(\omega)+\cdots+y_n(\omega)x_n(\omega)\Big)^2.$$

1) $\|\ \|_{\Omega}$ means the norm in $L_2(\Omega)$.

This lemma is an extension of Kolmogoroff's inequality, and its proof can be achieved in the same way and so will be omitted.

Let s_i, $i=0, 1, 2, \dots$, be any sequence dense in [0,1]. Any function $f(t, \omega) \in S(t_0, t_1, \dots, t_n)$ may be considered as an element in $S(t_0^{(m)}, t_1^{(m)}, \dots, t_{m+n+1}^{(m)})$, $t_0^{(m)}, t_1^{(m)}, \dots, t_{m+n+1}^{(m)}$ being the sequence $s_0, s_1, \dots, s_m, t_0, t_1, \dots t_n$, rearranged in the order of magnitude.
We obtain by the above lemma 2.1

$$P\{\omega ; \max_{0 \leq i \leq m+n+1} | I(t_i^{(m)}, \omega ; f) | \geqq b\} \leqq \frac{1}{b^2} \mathscr{E} I(1, \omega ; f)^2 = \frac{1}{b^2} \| f(\tau, \omega) \|_{[0,1] \times \Omega}^2 ,$$

a fortiori

$$P\{\omega ; \max_{0 \leq i \leq m} | I(s_i, \omega ; f) | \geqq b\} \leqq \frac{1}{b^2} \| f(\tau, \omega) \|_{[0,1] \times \Omega}^2 .$$

and so, as $m \to \infty$, we have

$$P\{\omega ; \sup_i | I(s_i, \omega, f) | \geqq b\} \leqq \frac{1}{b^2} \| f(\tau, \omega) \|_{[0,1] \times \Omega}^2 ,$$

which implies (B) on account of (C).

Theorem 2.2. There exists an extension of $I(t, \omega ; f)$ defined for any $f \in \bar{S}$ which satisfy (L), (N), (C), (I), (B) and (J). If $I_1(t, \omega ; f)$, $I_2(t, \omega ; f)$ be such extensions, then $I_1(t, \omega ; f) = I_2(t, \omega ; f)$ for any t with P-measure 1 for any $f \in \bar{S}$.

Definition. The function $I(t, \omega ; f)$ determined up to P-measure 0 *in the above theorem is called the (stochastic) integral of f with respect to $g(t, \omega)$ and is denoted by $\int_0^t f(\tau, \omega) dg(\tau, \omega)$.*

Proof of Theorem 2.2. Existence. $I(1, \omega ; f)$ is a linear operation from $S\left(\leq L_2([0, 1] \times \Omega)\right)$ to $L_2(\Omega)$, which is isometric on account of (I). We can extend $I(1, \omega ; f)$ and define a linear isometric operation from $\bar{S}$ to $L_2(\Omega)$. The extension is determined up to P-measure 0 for each $f(t, \omega) \in \bar{S}$. We denote it by $\tilde{I}(1, \omega ; f)$. Similarly for any t we can define $\tilde{I}(t, \omega ; f)$ which satisfy (L), (N) and (I).

Let $f_n(t, \mathrm{w})$ be a sequence in S such that

$$\| f_{n+1} - f_n \|_{[0,1] \times \Omega}^2 \leqq \frac{1}{8^n} . \tag{2.4}$$

By (B) we obtain

$$P\left\{\omega ; \sup_{0 \leq t \leq 1} | I(t, \omega ; f_{n+1}) - I(t, \omega ; f_n) | \geqq \frac{1}{2^n} \right\} \leqq \frac{1}{2^n} \tag{2.5}$$

By Borel-Contelli's theorem we have

$$\sup_{0 \leq t \leq 1} | I(t, \omega ; f_{n+1}) - I(t, \omega ; f_n) | < \frac{1}{2^n} \tag{2.6}$$

for a sufficiently large number n with P-measure 1. Therefore $\{I(t, f_n ; \omega)\}$ will be convergent uniformly in t with P-measure 1. Denote the limit by $I(t, \omega ; f)$.

$\tilde{I}(t, \omega ; f)$, as the $\| \quad \|_{\Omega}$-limit of the sequence $I(t, \omega ; f_n)$, is also the limit of a subsequence of $\{I(t, \omega; f_n)\}$, (in the truth "of the sequence itself") with P-measure 1 for any t. Thus we have

$$(2.7) \qquad P\{\omega;\ I(t, \omega; f) = \tilde{I}(t, \omega; f)\} = 1.$$

Now we shall verify the properties (L), (N), (C), (I), (B) and (J) for this extension $I(t, \omega; f)$. (N) is clear. $I(t, \omega; f_n)$ being continuous with P-measure 1 by (C), $I(t, \omega; f)$ will also satisfy (C). Since (L) and (I) hold for $\tilde{I}(t, \omega; f)$, it is also the case with $I(t, \omega; f)$. For the proof of (B) we make use of the above-cited sequence $\{f_n\}$. We have clearly by (B) $P\{\omega; \sup_{0 \leq t \leq 1} |I(t, \omega; f_n)| \geqq b\} \leqq \frac{1}{b^2} \|f_n\|^2_{[0,1]\times\Omega}$. As $n \to \infty$, we obtain (B), for $\{I(t, \omega; f_n)\}$ converges to $I(t, \omega; f)$ uniformly in t with P-measure 1, while $\{f_n(t, \omega)\}$ $\|\ \|_{[0,1]\times\Omega}$- converges to $f(t, \omega)$.

In order to prove (J) we need only prove that $I(t, \omega; f) = I(t, \omega; h)$ almost everywhere in Ω_1 for each value of t, because $I(t, \omega; f)$ and $I(t, \omega; g)$ are continuous in t with P-measure 1 on account of (C). Let $f_n(t, \omega)$, $h_n(t, \omega)$, $n = 1, 2, \dots$ be sequences in S such that

$$(2.8) \qquad \|f_n - f\|^2_{[0,1]\times\Omega} \leqq \frac{1}{8^n}, \qquad \|h_n - h\|^2_{[0,1]\times\Omega} \leqq \frac{1}{8^n}.$$

Define $k_n(t, \omega)$, by

$$(2.9) \qquad \begin{aligned} k_n(t, \omega) &= f_n(t, \omega) \text{ for } \omega \in \Omega_1, \\ &= h_n(t, \omega) \text{ for } \omega \in \Omega - \Omega_1. \end{aligned}$$

Then we have

$$(2.10) \qquad \|k_n - h\|^2_{[0,1]\times\Omega} \leqq \frac{2}{8^n}.$$

By (2.8) and (2.10) we obtain

$$\|I(t, \omega; f_n) - I(t, \omega; f)\|^2_\Omega \leqq \frac{1}{8^n},$$

$$\|I(t, \omega; k_n) - I(t, \omega; h)\|^2_\Omega \leqq \frac{2}{8^n}.$$

By the use of Bienaymé's inequality and Borel-Cantelli's theorem we see that $\{I(t, \omega; f_n)\}$ and $\{I(t, \omega; k_n)\}$ converge to $I(t, \omega; f)$ and to $I(t, \omega; h)$ respectively with P-measure 1. Since (J) holds in S, we have $I(t, \omega; f_n) = I(t, \omega; k_n)$ almost everywhere in Ω_1, and so $I(t, \omega; f) = I(t, \omega; k)$ almost everywhere in Ω_1.

Uniqueness. Let $\{f_n(t, \omega)\}$ be any seqnence in S, $\|\ \|_{[0,1]\times\Omega}$-convergent to $f(t, \omega)$ $(\in S)$. Let $I_1(t, \omega; f)$ and $I_2(t, \omega; f)$ be two extensions. By (I), $I_1(t, \omega; f_n)$ and $I_2(t, \omega; f_n)$ $\|\ \|_\Omega$-converge to $I_1(t, \omega; f)$ and to $I_2(t, \omega; f)$ respectively. Therefore we have $I_1(t, \omega; f) = I_2(t, \omega; f)$ with P-measure 1 for any t, and so $I_1(t, \omega; f) = I_2(t, \omega; f)$ for any t with P-measure 1 on account of (C).

At last we shall define $I(t, \omega; f)$ *for* $f(t, \omega) \in S^*$. We choose $f_n(t, \omega) \in S$, $n = 1, 2, \dots$, such that $P\{\omega; f_n(t, \omega) = f(t, \omega)\} > 1 - \varepsilon$, Write Ω_n for the ω-set in { }. We define $I(t, \omega; f)$ as $I(t, \omega; f_n)$ on $\Omega_n - \bigcup_{k=1}^{n-1} \Omega_k$. Thus we can define $I(t, \omega; f)$ on the set $\bigcup_n \Omega_n$ of P-measure 1. This definition is independent of its procedure on account of (J), and we can easily verify the properties (L), (N), (C), and (J) for this integral.

3. We shall show important subclasses of S or of S^*.

Theorem 3.1. $L_2([0, 1]) \leq \bar{S}$[1]

The proof is brief and so will be omitted. By this theorem we see that our integral is an extension of that of Paley and Wiener.

Theorem 3.2. Any bounded function $f(t, \omega)$ in G belongs to $\bar{S}$.

Proof. Let M denote an upper bound of $|f(\tau, \omega)|$. We shall define $f(t, \omega)=0$ in the case: $t<0$. Then it holds that $f(\tau, \omega)$, $\tau \leqq t$ and $g(\tau, \omega)$, $0 \leqq \tau \leqq t$ are independent of $g(\sigma, \omega)-g(t, \omega)$, $t \leqq \sigma \leqq 1$ for $0 \leqq t \leqq 1$. Define $\psi_n(t)$ by $\psi_n(t)=(k-1)2^{-n}$ if $(k-1)2^{-n} \leqq t < k2^{-n}$. By (the slight modification of) Doob's Lemma[2] there exist a number c and a sequence of integers a_n such that $\lim_{n\to\infty} f\big(\psi_{a_n}(t-c)+c, \omega\big)=f(t, \omega)$ almost everywhere in Ω. Put $f_n(t, \omega)=f\big(\psi_{a_n}(t-c)+c, \omega\big)$. Since we have $\psi_{a_n}(t-c)+c \leqq t$ by the definition, $f_n(t, \omega)$ belongs to G, and since $|f_n(t, \omega)| \leqq M$, we have $f_n(t, \omega) \in L^2_{[0,1]\times\Omega}$. Therefore we have $f_n(t, \omega) \in S$ by the definition. Since $|f_n(t, \omega)| \leqq M$, $n=1, 2, \dots$, and $\{f_n(t, \omega)\}$ converges to $f(t, \omega)$ almost everywhere, $\{f_n(t, \omega)\}$ will $\| \ \|_{[0,1]\times\Omega}$-converge to $f(t, \omega)$.

Theorem 3.3. If any function $f(t, \omega) \in G$ is P-measurable in ω for any t and is a function of t continuous except possibly for discontinuities of the first kind[3] *with P-measure 1, then $f(t, \omega)$ belongs to S^*.*

Proof. It is clear that $\sup_{0\leq t\leq 1} f(t, \omega)$ is equal to $\sup f(t, \omega)$ for t running over all rational numbers in [0, 1] with P-measure 1. Denote it by $M(\omega)$. Then $M(\omega)$ is measurable in ω and is finite with P-measure 1.

For any ε we determine N such that $P\{\omega; M(\omega)<N\}>1-\varepsilon$. Define $f_N(t, \omega)$ as $f(t, \omega)$ on this ω-set in { } and as 0 otherwise. Then $f_N(t, \omega)$ is a measurable (in t, ω) bounded function $\in G$, and so we have $f_N(t, \omega) \in \bar{S}$. On the other hand we have

$$P\{\omega; f(t, \omega)=f_N(t, \omega)\}=P\{\omega; M(\omega)<N\}>1-\varepsilon. \quad \text{q.e.d.}$$

4. *Example.* Let $F(x)$ be a function of x such that $F''(x)$ may be continuous. By Theorem 3.3 we see $F'\big(g(t, \omega)\big) \in S^*$ The author has proved the equality[4]:

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1) Any function of t can also be considered as a function of (t, ω). In this sense $L_2(\Omega)$ will be considered as a subset of $L_2([0, 1]\times\Omega)$.

2) J, L. Doob. Loc. cit. p. 512 (*1*) Lemma *2.1.*

3) $f(t)$ is called to have a discontinuity of the first kind at t, if $f(t+0)$ and $f(t-0)$ exist and $f(t+0)=f(t) \neq f(t-0)$.

4) Cf. K. Itô: 'Markoff 過程ヲ定メル微分方程式 § 7. 全國紙上數學談話會 第 244 號.

$$\int_0^t F'\big(g(\tau, \omega)\big)d_\tau g(\tau, \omega) = F\big(g(t, \omega)\big) - F\big(g(0, \omega)\big)$$
$$-\frac{1}{2}\int_0^t F''\big(g(\tau, \omega)\big)d\tau .$$

In the last term we may see a characteristic property by which we distinguish "stochastic integral" from "ordinary integral."