143. Two-dimensional Brownian Motion and Harmonic Functions.

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1. The purpose of this paper is to investigate the properties of two-dimensional Brownian motions[1] and to apply the results thus obtained to the theory of harmonic functions in the Gaussian plane. Our starting point is the following theorem: *Let D be a domain in the Gaussian plane R^2, and let E be a closed set on the boundary $Bd(D)$ of D. Then, under certain assumptions on D and E, the probability $P(\zeta, E, D)$, that the Brownian motion starting from a point $\zeta \in D$ will enter into E without entering into the other part $Bd(D)-E$ of the boundary of D before it, is equal to the harmonic measure in the sense of R. Nevanlinna of E with respect to the domain D and the point ζ.*

It is expected that, by means of this method, many of the known results in the theory of harmonic or analytic functions will be interpreted from the standpoint of the theory of probability. We shall here give only the fundamental results and a few of its applications, leaving the detailed discussions of further applications to another occasion.

Most of the results obtained in this paper are also valid for the case of higher dimensional Brownian motions. But there are also many theorems in which the dimension number plays an essential rôle[3]. For example, Theorems 6, 7 and 8 of this paper are no longer true in R^3. The situation will become clearer if we observe the following theorem: *Consider the n-dimensional Brownian motion in $R^n (n \geqq 2)$, and let $\bar{K}^n$ be the closed unit sphere in R^n. Then, for any $\zeta \in R^n - \bar{K}^n$, the probability $P(\zeta, \bar{K}^n, R^n - \bar{K}^n)$ that the Brownian motion starting from ζ will enter into $\bar{K}^n$ for some $t > 0$ is equal to $|\zeta|^{2-n}$ if $n \geqq 3$*[4]*, while this probability is $\equiv 1$ on $R^n - \bar{K}^n$ if $n = 2$*[5]. This result is closely related with the fact that there is no bounded harmonic function, other than the constant 1, which is defined on $R^n - \bar{K}^n$ and tends to 1 as $|\zeta| \to 1$, while, for any $n \geqq 3$, $u(\zeta) = |\zeta|^{2-n}$ is a non-trivial example of a bounded harmonic function with the said property.

2. Let $\{z(t, \omega) = \{x(t, \omega), y(t, \omega)\} \mid -\infty < t < \infty, \omega \in \Omega\}$ be a two-dimensional Brownian motion defined on the $z = \{x, y\}$-plane R^2, i.e. an independent system of two one-dimensional Brownian motions $\{x(t, \omega) \mid$

1) Brownian motions were discussed by N. Wiener and P. Lévy. Cf. N. Wiener, Generalized harmonic analysis, Acta Math., **54** (1930); N. Wiener, Homogeneous chaos. Amer. Journ. of Math., **60** (1939); R. E. A. C. Paley and N. Wiener, Fourier transforms in the complex domain, New York, 1933; P. Lévy, L'addition des variables aléatoires, Paris, 1937; P. Lévy, Sur certains processus stochastiques homogènes, Compositio Math., **7** (1939); P. Lévy, Le mouvement brownien plan, Amer. Journ. of Math., **61** (1940).

2) Cf. R. Nevanlinna, Eindeutige analytische Funktionen, Berlin, 1937.

3) Cf. S. Kakutani, On Brownian motions in n-space, Proc. **20** (1944).

4) $|\zeta|$ denotes the Euclidean distance of ζ from the origin of R^n.

5) The case $n=2$ is contained in Theorem 4 of this paper.

$-\infty<t<\infty, \omega\in\Omega\}$ and $\{y(t,\omega)\,|-\infty<t<\infty, \omega\in\Omega\}$[1]. It is easy to see that this definition is independent of the choice of rectangular coordinate systems in R^2. Further, it is known that there exists a null set N in Ω such that, for any $\omega\in\Omega-N$, $z(t,\omega)$ is a complex-valued continuous function of t.

Let F be a closed subset of R^2. For any $\zeta\in R^2-F$ and for any $\omega\in\Omega-N$, let us consider the path of a moving point $\zeta+\big(z(t,\omega)-z(0,\omega)\big)$, $t\geqq 0$, starting from ζ. Then either (i) there exists a positive number $\tau=\tau(\zeta,F,\omega)$ such that $\zeta+\big(z(\tau,\omega)-z(0,\omega)\big)\in F$ and $\zeta+\big(z(t,\omega)-z(0,\omega)\big)\in R^2-F$ for any t with $0<t<\tau$, or (ii) $\zeta+\big(z(t,\omega)-z(0,\omega)\big)\in R^2-F$ for any t with $t>0$. The set of all $\omega\in\Omega-N$ for which the first case happens is denoted by $\Omega(\zeta,F)$. It is clear that, for any $\omega\in\Omega(\zeta,F)$, $\tau(\zeta,F,\omega)$ denotes the time when the moving point $\zeta+\big(z(t,\omega)-z(0,\omega)\big)$ starting from ζ enters into F for the first time after $t=0$[2].

Lemma 1. $\Omega(\zeta,F)$ is a measurable subset of Ω, and $\tau(\zeta,F,\omega)$ is a real-valued measurable function of ω on $\Omega(\zeta,F)$.

Proof. It suffices to show that the set $\{\omega\,|\,\tau(\zeta,F,\omega)>\sigma\}$ is measurable for any positive number σ, and this follows from the relation

$$(1)\qquad \{\omega\,|\,\tau(\zeta,F,\omega)>\sigma\}$$
$$=\Big\{\omega\,|\,\zeta+\big(z(t,\omega)-z(0,\omega)\big)\in R^2-F \text{ for any } t \text{ with } 0\leqq t\leqq\sigma\Big\}$$
$$=\bigcup_{n=1}^{\infty}\Big\{\omega\,\Big|\,d\Big(\zeta+\big(z(t,\omega)-z(0,\omega)\big),F\Big)\geqq\frac{1}{n} \text{ for any } t \text{ with } 0\leqq t\leqq\sigma\Big\}^{3)}$$
$$=\bigcup_{n=1}^{\infty}\bigcap_{0\leq r_i\leq\sigma}\Big\{\omega\,\Big|\,d\Big(\zeta+\big(z(r_i,\omega)-z(0,\omega)\big),F\Big)\geqq\frac{1}{n}\Big\},$$

where $\bigcap_{0\leq r_i\leq\sigma}$ means that the intersection is taken for all rational numbers r_i satisfying $0\leqq r_i\leqq\sigma$.

Let us further put

$$(2)\qquad \alpha(\zeta,F,\omega)=z\big(\tau(\zeta,F,\omega),\omega\big)$$

for any $\omega\in\Omega(\zeta,F)$. It is easy to see that, for any $\omega\in\Omega(\zeta,F)$, $\alpha(\zeta,F,\omega)$ is the point at which the moving point $\zeta+\big(z(t,\omega)-z(0,\omega)\big)$ starting from ζ enters into F for the first time after $t=0$.

Lemma 2. $\alpha(\zeta,F,\omega)$ is a complex-valued measurable function of ω on $\Omega(\zeta,F)$.

Proof. It suffices to show that, for any closed subset E of F, the

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1) Cf. S. Kakutani, loc. cit. 3) p. 700.

2) Put (i) $\tau(\zeta,F,\omega)=\infty$ if $\zeta\in R^2-F$ and $\omega\in\Omega-N-\Omega(\zeta,F)$, (ii) $\tau(\zeta,F,\omega)=0$ if $\zeta\in F$ and $\omega\in\Omega-N$. Then $\tau(\zeta,F,\omega)$ is defined for any $\zeta\in R^2$ and $\omega\in\Omega-N$. In case $\zeta\in F$, there is another function $\tau'(\zeta,F,\omega)$ which is useful in some problems: $\tau'(\zeta,F,\omega)$ is the inf of all $t>0$ such that $\zeta+\big(z(t,\omega)-z(0,\omega)\big)\in F$. If there is no such $t>0$, then $\tau'(\zeta,F,\omega)=\infty$.

3) $d(\zeta,F)=\inf_{\zeta'\in F}|\zeta-\zeta'|$ is the distance of ζ from F.

set $\{\omega \mid \alpha(\zeta, F, \omega) \in E\}$ is a measurable subset of $\Omega(\zeta, F)$ and this follows from Theorem 1 if we observe that

$$(3) \qquad \{\omega \mid \alpha(\zeta, F, \omega) \in E\} = \{\omega \mid \tau(\zeta, F, \omega) = \tau(\zeta, E, \omega)\}.$$

3. Let F be a closed subset of R^2. For any $\zeta \in R^2 - F$, let $D = D(\zeta)$ be the component of $R^2 - F$ which contains ζ. Then the boundary $Bd(D)$ of D is a closed subset of F and it is clear that

$$(4) \qquad \tau(\zeta, F, \omega) = \tau\big(\zeta, Bd(D), \omega\big),$$

$$(5) \qquad \alpha(\zeta, F, \omega) = \alpha\big(\zeta, Bd(D), \omega\big)$$

for any $\omega \in \Omega(\zeta, F)$.

Let us finally put

$$(6) \qquad P(\zeta, E, D) = Pr\Big\{\omega \mid \alpha\big(\zeta, Bd(D), \omega\big) \in E\Big\}$$

for any closed subset E of the boundary $Bd(D)$ of D. This is the probability that the Brownian motion starting from ζ will enter into E for some $t > 0$ without entering into $Bd(D) - E$ before it.

$P(\zeta, E, D)$ may also be defined for any Borel subset of $Bd(D)$. Further, for any Borel subset E of R^2, we may put $P(\zeta, E, D) = P(\zeta, E \frown Bd(D), D)$. If $E \supset Bd(D)$, or in particular if $E = F$, where F is a closed subset of R^2 from which $D = D(\zeta)$ is defined as the component of $R^2 - F$ which contains ζ, then $P(\zeta, E, D) = P\big(\zeta, Bd(D), D\big)$ is equal to the probability $Pr\{\Omega(\zeta, F)\}$ that the Brownian motion starting from ζ will enter into F for some $t > 0$. It is easy to see that

Lemma 3. $P(\zeta, Bd(D), D) \equiv 1$ *on* D *if* D *is a bounded domain.*

We can prove that $Pr\{\Omega(\zeta, F)\} = 1$ for any $\zeta \in R^2 - F$ if F is a closed set containing a continuum. It will be shown later (Theorem 7) that this probability is $\equiv 0$ or $\equiv 1$ on $R^2 - F$ according as F is of zero or of positive capacity[1].

4. Under a *Jordan domain* we understand a finitely connected domain in R^2 whose boundary $Bd(D)$ consists of a finite number of simple closed Jordan curves disjoint from one another. In case D is bounded, there exists one and only one simple closed Jordan curve among these which separates every point of D from ∞. This curve is called the *outer boundary* of D and is denoted by $OBd(D)$. (If D is not bounded, $OBd(D)$ is understood to be empty). The remainder $Bd(D) - OBd(D)$ is called the *inner boundary* of D and is denoted by $IBd(D)$. $IBd(D)$ is either empty, in which case D is simply connected, or consists of a finite number of mutually disjoint simple closed Jordan curves.

A subset E of the boundary $Bd(D)$ of a Jordan domain D is an *elementary set* if it consists of a finite number of mutually disjoint non-abutting Jordan arcs on $Bd(D)$, in- or excluding the end points. It is clear that E is an elementary subset of $Bd(D)$ if and only if

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1) As for the notion of capacity, cf. R. Nevanlinna, loc. cit. 2). Usually capacity is defined only for compact sets. The capacity of a closed unbounded set F is defined as the supremum of the capacities of all compact subsets of F.

$Bd(D)-E$ is so. Further, $\zeta \in E$ is an *inner point* of an elementary set E on $Bd(D)$ if it is not an end point of an arc constituting E.

Theorem 1. Let D be a Jordan domain. We do not assume that D is bounded or simply connected. Let further E be an elementary set on the boundary $Bd(D)$ of D. Then the probability $P(\zeta, E, D)$, that the Brownian motion starting from a point $\zeta \in D$ will enter into E for some $t>0$ without entering into $Bd(D)-E$ before it, is a bounded harmonic function of ζ in D and

$$\lim_{\zeta \in D, \zeta \to \zeta_0} P(\zeta, E, D)=1 \text{ or } 0 \tag{7}$$

according as ζ_0 is an inner point of E or of $Bd(D)-E$.

Proof. We shall first discuss the case when D is bounded. In order to prove the first proposition, it suffices to show that

$$\frac{1}{2\pi}\int_0^{2\pi} P(\zeta_0+re^{i\theta}, E, D)d\theta = P(\zeta_0, E, D) \tag{8}$$

for any $\zeta_0 \in D$ and for any $r>0$ such that the circular domain $K(\zeta_0, r) =\{\zeta \mid |\zeta-\zeta_0|<r\}$ is entirely contained in D (i.e. contained in D together with its boundary $C(\zeta_0, r)=\{\zeta \mid |\zeta-\zeta_0|=r\}$). If A is an arc on the circumference $C(\zeta_0, r)$ and if we denote by $|A|$ the angular measure of A divided by 2π, then the direction homogeneity of the two-dimensional Brownian motion will imply

$$P\big(\zeta_0, A, K(\zeta_0, r)\big)=|A|. \tag{9}$$

By using (9), (8) may be written as

$$\int_{C(\zeta_0, r)} P\big(\zeta_0, d\zeta, K(\zeta_0, r)\big)P(\zeta, E, D)=P(\zeta_0, E, D). \tag{10}$$

Since the Brownian motion is a temporally homogeneous differential process, the relation (10) is intuitively clear if we appeal to the notion of conditional probability. But in order to prove (10) rigorously we need to show that

$$\frac{Pr\{\widetilde{E} \cap \widetilde{A}\}}{Pr\{\widetilde{A}\}} \equiv \frac{Pr\{\widetilde{E} \cap A\}}{|A|} \to P(\zeta, E, D) \tag{11}$$

as A shrinks down to a point $\zeta \in C(\zeta_0, r)$, where

$$\widetilde{E}=\big\{\omega \mid \alpha(\zeta_0, Bd(D), \omega) \in E\big\}, \tag{12}$$

$$\widetilde{A}=\big\{\omega \mid \alpha(\zeta_0, C(\zeta_0, r), \omega) \in A\big\}. \tag{13}$$

This, however, requires a complicated argument, and so will not be discussed in this paper.

The second proposition of Theorem 1 is also intuitively clear; but it is not so easy to prove it in a rigorous way. We omit the proof.

In order to prove our theorem for an unbounded D, let $\{D_m \mid m =1, 2\dots\}$ be a sequence of bounded Jordan domains such that

$\bar{D}_m \subset D_{m+1} \subset D$, $OBd(D_m) \equiv C_m \subset D_{m+1}$, $IBd(D_m) = Bd(D)$, $m=1, 2, \dots$ and $\bigcup_{m=1}^{\infty} D_m = D$. Let us put

$$u_m(\zeta) = P(\zeta, E, D_m) \tag{14}$$

for any $\zeta \in D_m$, where E is an elementary set on $IBd(D_m) = Bd(D)$. Then it is easy to see that $u_{m+1}(\zeta) \geqq u_m(\zeta)$ on D_m, $m=1, 2, \dots$ Hence $\lim_{m\to\infty} u_m(\zeta) = u(\zeta)$ exists on $D = \bigcup_{m=1}^{\infty} D_m$ and the convergence is uniform on every compact set contained in D. It is not difficult to see that the limit function $u(\zeta)$, which is harmonic in D and is clearly independent of the choice of the sequence $\{D_m \mid m=1, 2, \dots\}$, is equal to the probability $P(\zeta, E, D)$ that the Brownian motion starting from ζ will enter into E for some $t > 0$ without entering into $Bd(D) - E$ before it. From this follows easily that $u(\zeta) = P(\zeta, E, D)$ has the properties as stated in Theorem 1.

5. We shall give some applications of Theorem 1.

Theorem 2. Let F be a closed set in R^2 with an inner point. Then, for any $\zeta \in R^2 - F$, the probability $Pr\{\Omega(\zeta, F)\}$ that the Brownian motion starting from ζ will enter into F for some $t > 0$ is 1.

Proof, It suffices to discuss the case when F is a closed circular domain $\overline{K(\zeta_0, r)} = \{\zeta \mid |\zeta - \zeta_0| \leqq r\}$. From Theorem 1 we see that $Pr\left\{\Omega\left(\zeta, \overline{K(\zeta_0, r)}\right)\right\}$ is a bounded harmonic function of ζ defined on $R^2 - \overline{K(\zeta_0, r)}$ which tends to 1 as $|\zeta - \zeta_0| \to r$. Since there is no such function other than the constant 1, this proves our theorem.

Remark. Lemma 3 is a special case of Theorem 2.

Theorem 3. Let D be a Jordan domain. We do not assume that D is bounded or simply connected. Let $f(\zeta)$ be a real-valued continuous function defined on the boundary $\Gamma \equiv Bd(D)$ of D. Then, for any $\zeta_0 \in D$, the value $u(\zeta_0)$ of the solution $u(\zeta)$ of the Dirichlet problem for the domain D and the boundary values $f(\zeta)$ is obtained by taking the integral of a Poisson type of $f(\zeta)$ with respect to the kernel $P(\zeta_0, E, D)$ on Γ or by taking the mathematical expectation of the composed function $f\left(\alpha(\zeta_0, \Gamma, \omega)\right)$:

$$u(\zeta_0) = \int_\Gamma P(\zeta_0, d\zeta, D) f(\zeta) = \int_\Omega f\left(\alpha(\zeta_0, \Gamma, \omega)\right) d\omega^{1)}. \tag{15}$$

In Theorem 3 D can be any domain regular for the Dirichlet problem. The assumption that D is a Jordan domain is not essential. Further, even if D is not regular, we may still get a *generalized solution* of the Dirichlet problem in this way. The discussions of such general cases are left to another paper.

Theorem 4. Let D_1 and D_2 be two Jordan domains such that $D_1 \subset D_2$ and $Bd(D_1) \frown Bd(D_2) \neq \theta$. If E is an elementary set contained in $Bd(D_1) \frown Bd(D_2)$, then $P(\zeta, E, D_1) \leqq P(\zeta, E, D_2)$ for any $\zeta \in D_1$, where the equality holds if and only if $D_1 = D_2$.

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1) $\alpha(\tau, \Gamma, \omega)$, and hence $f\left(\alpha(\zeta, \Gamma, \omega)\right)$, is defined only on $\Omega(\zeta, F)$. But since $\Omega - \Omega(\zeta, F)$ is a null set in our case, so we may speak of the integral of $f\left(\alpha(\zeta, \Gamma, \omega)\right)$ on Ω.

In the theory of harmonic functions this theorem is known as the *principle of the extension of domain.* In our argument this follows immediately from the definition of $P(\zeta, E, D)$.

6. *Theorem* 5. *Let F be a compact subset of R^2 such that the complementary R^2-F is connected.* (*For example, we may take as F any totally disconnected compact set in R^2.*) *Let D be a simply connected bounded*[1] *Jordan domain which contains F, and observe a function $P(\zeta, F, D-F)$*[2] *of ζ defined in $D-F$. Then $P(\zeta, F, D-F)$ coincides with the harmonic measure in the sense of R. Nevanlinna*[3] *of F with respect to the domain $D-F$ and the point ζ. If F is of zero capacity, then $P(\zeta, F, D-F)\equiv 0$ on $D-F$. If F is of positive capacity, then $P(\zeta, F, D-F)$ is harmonic and >0 in $D-F$. Further,* (i) *$P(\zeta, F, D-F)\to 0$ as $\zeta\to\zeta_0\in Bd(D)$.* (ii) *if $\zeta_0\in F$ and if F is locally of zero capacity at ζ_0 (i. e. if there exists an $r_0>0$ such that $F\frown\overline{K(\zeta_0, r_0)}$*[4] *is of zero capacity), then $P(\zeta, F, D-F)$ can be extended to a function harmonic and <1 in $(D-F)\frown K(\zeta_0, r_0)$.* (iii) *if $\zeta_0\in F$ and if F is locally of positive capacity at ζ_0 (i. e. if $F\frown\overline{K(\zeta_0, r)}$ is of positive capacity for any $r>0$), then $\sup_{\zeta\in(D-F)\frown K(\zeta_0, r_0)}P(\zeta, F, D-F)=1$ for any $r>0$.*

Proof. Let $\{\varDelta_n \mid n=1, 2, \dots\}$ be a sequence of bounded Jordan domains such that $\varDelta_1\subset D$, $\overline{\varDelta}_n\subset\varDelta_{n+1}$, $OBd(\varDelta_n)=Bd(D)\equiv C$, $IBd(\varDelta_n)\equiv \Gamma_n\subset D_{n+1}$, $n=1, 2, \dots$ and $\bigcup_{n=1}^{\infty}\varDelta_n=D-F$.

Let us put

$$v_n(\zeta)=P(\zeta, \Gamma_n, \varDelta_n)\,, \tag{16}$$

for $n=1, 2, \dots$ This is the probability that the Brownian motion starting from ζ will enter into Γ_n for some $t>0$ without entering into $Bd(\varDelta_n)-\Gamma_n=Bd(D)\equiv C$ before it. Thus each $v_n(\zeta)$ is bounded and harmonic in $\varDelta_n$ and $v_n(\zeta)\to 0$ or 1 according as $\zeta\to\zeta_0\in C$ or $\zeta\to\zeta_0\in\Gamma_n$.

It is clear that $v_{n+1}(\zeta)\leqq v_n(\zeta)$ in $\varDelta_n$, $n=1, 2, \dots$ Hence $\lim_{n\to\infty}v_n(\zeta)=v(\zeta)$ exists in $D-F=\bigcup_{n=1}^{\infty}\varDelta_n$, and the convergence is uniform on every compact set contained in $D-F$. The limit function $v(\zeta)$, which is bounded and harmonic in $D-F$ and is clearly independent of the choice of the sequence $\{\varDelta_n \mid n=1, 2, \dots\}$, is nothing else than the harmonic measure in the sence of R. Nevanlinna of F with respect to the domain $D-F$ and the point ζ.

On the other hand, from the definition of $v_n(\zeta)$, it is easy to see that, for any $\zeta\in D-F$, the limit function $v(\zeta)$ is equal with the probability $P(\zeta, F, D-F)$ that the Brownian motion starting from ζ will enter into F for some $t>0$ without entering into $Bd(D-F)-F=Bd(D)\equiv C$ before it. This completes the proof of the first proposition of Theorem 6. The rest of the theorem then follows easily from the

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1) It is inessensial that D is bounded or simply connected.

2) $P(\zeta, F, D-F)$ is defined as the probability that the Brownian motion starting from $\zeta\in D-F$ will enter into F for some $t>0$ without entering into $Bd(D-F)-F=Bd(D)$ before it.

3) Cf. R. Nevanlinna, Eindeutige analytische Funktion, Berlin, 1936.

4) $K(\zeta_0, r_0)=\{\zeta \mid |\zeta-\zeta_0|<r_0\}$ is a circular domain with the center ζ_0 and the radius r_0 $\overline{K(\zeta_0, r_0)}$ is the closure of $K(\zeta_0, r)$ so that $\overline{K(\zeta_0, r_0)}=\{\zeta \mid |\zeta-\zeta_0|\leqq r\}$.

well-known properties of harmonic measure and capacity.

Remark. It is quite natural to expect that, if $\zeta_0 \in F$ and if F is locally of positive capacity at ζ_0, then $P(\zeta, F, D-F) \rightarrow 1$ as $\zeta \rightarrow \zeta_0$. But it may be shown by an example that this is not always the case.

7. *Theorem 6. Let F be a compact set in R^2. Then the probability $Pr\{\Omega(\zeta, F)\}$ that the Brownian motion starting from $\zeta \in R^2 - F$ will enter into F for some $t > 0$ is $\equiv 0$ or $\equiv 1$ on $R^2 - F$ according as F is of zero or of positive capacity.*

Proof. Let us first discuss the case when $R^2 - F$ is not connected. As is well known,[1] such a case can happen only when F is of positive capacity. Let $\zeta \in R^2 - F$, and let $D = D(\zeta)$ be the component of $R^2 - F$ which contains ζ. If D is bounded, then Lemma 3 implies that $Pr\{\Omega(\zeta, F)\} = P\big(\zeta, Bd(D), D\big) = 1$. If D is not bounded, then D is the only component of $R^2 - F$ which extends to ∞, and hence $Pr\{\Omega(\zeta, F)\} = Pr\{\Omega(\zeta, F')\}$, where $F' \equiv R^2 - D$ is a compact set containing F as a subset such that the complementary $R^3 - F' = D$ is connected. Thus we see that it suffices to discuss the case when $R^2 - F$ is connected.

Let now F be a compact set in R^2 such that the complementary $R^2 - F$ is connected. Let $\{D_m \mid m = 1, 2, \ldots\}$ be a sequence of simply connected bounded Jordan domains such that $F \subset D_1$, $\overline{D}_m \subset D_{m+1}$, $m = 1, 2, \ldots$ and $\bigcup_{m=1}^{\infty} D_m = R^2$. For each m, put

$$(17) \qquad w_m(\zeta) = P(\zeta, F, D_m - F),$$

where $P(\zeta, F, D_m - F)$ is a harmonic function function of ζ defined on $D_m - F$ discussed in § 6. It is easy to see that $w_{m+1}(\zeta) \geqq w_m(\zeta)$ on $D_m - F$, $m = 1, 2, \ldots$ Hence $\lim_{m\to\infty} w_m(\zeta) = w(\zeta)$ exists on $R^2 - F = \bigcup_{m=1}^{\infty}(D_m - F)$, and the convergence is uniform on every compact set contained in $R^2 - F$. The limit function $w(\zeta)$ is harmonic in $R^2 - F$ and is clearly independent of the choice of the sequence $\{D_m \mid m = 1, 2, \ldots\}$. Further, it is not difficult to see that $w(\zeta)$ is equal to the probability $P(\zeta, F, R^2 - F) = Pr\{\Omega(\zeta, F)\}$ that the Brownian motion starting from ζ will enter into F for some $t > 0$.

It is clear that $w(\zeta) = P(\zeta, F, R^2 - F) \equiv 0$ on $R^2 - F$ if F is of zero capacity. This follows from the fact that $P(\zeta, F, D - F) \equiv 0$ on $D_m - F$ for $m = 1, 2, \ldots$ Let us assume that F is of positive capacity. Let $\{\Delta_n \mid n = 1, 2, \ldots\}$ be a sequence of bounded Jordan domains with the properties, as stated in the proof of Theorem 5, with respect to the compact set F and the simply connected bounded Jordan domain D_1. Let $D_{m,n}$ be the bounded Jordan domain such that $OBd(D_{m,n}) = Bd(D_m) \equiv C_m$ and $IBd(D_{m,n}) = IBd(\Delta_n) \equiv \Gamma_n$, $m, n = 1, 2, \ldots$ It is easy to see that $D_{m',n'} \supset D_{m,n}$ if $m' \geqq m$ and $n' \geqq n$, and further that $\bigcup_{m,n=1}^{\infty} D_{m,n} = R^2 - F$. For any $\zeta \in D_{m,n}$ and for any elementary set E on the boundary $Bd(D_{m,n}) = C_m \cup \Gamma_n$ of $D_{m,n}$, let us consider a function $P(\zeta, E, D_{m,n})$ of ζ defined on $D_{m,n}$, and put

$$(18) \qquad v_{m,n}(\zeta) = P(\zeta, \Gamma_n, D_{m,n}).$$

1) In order that a compact set F be of zero capacity it is necessary that it is totally disconnected.

$v_{m,n}(\zeta)$ is a bounded harmonic function defined on $D_{m,n}$ which tends to 0 or 1 according as ζ tends to a point belonging to C_m or Γ_n, and so denotes the probability that the Brownian motion starting from ζ will enter into Γ_n for some $t>0$ without entering into C_m before it. It is clear that

$$(19) \qquad \lim_{n\to\infty} v_{m,n}(\zeta)=w_m(\zeta)$$

on D_m-F, and that the convergence is uniform on every compact set contained in D_m-F.

For any $\zeta_0 \in R^2-F$, let n_0 be so large that $\zeta_0 \in D_{m,n}$ for $m, n \geqq n_0$. Since $w(\zeta)$ is harmonic on R^2-F, so we see from Theorem 3 that

$$(20) \qquad v(\zeta_0)=\int_{\Gamma_n} P(\zeta_0, d\zeta, D_{m,n})w(\zeta)+\int_{C_m} P(\zeta_0, d\zeta, D_{m,n})w(\zeta)\,.$$

Let $n\to\infty$ in (20). Then the first integral on the right hand side tends to $v_m(\zeta_0)$ since

$$(21) \qquad \int_{\Gamma_n} P(\zeta_0, d\zeta, D_{m,n})w(\zeta) \geqq \int_{\Gamma_n} P(\zeta_0, d\zeta, D_{m,n})w_m(\zeta)=w_m(\zeta_0)$$

and

$$(22) \qquad \int_{\Gamma_n} P(\zeta_0, d\zeta, D_{m,n})w(\zeta) \leqq \int_{\Gamma_n} P(\zeta_0, d\zeta, D_{m,n})=P(\zeta_0, \Gamma_n, D_{m,n})$$
$$=v_{m,n}(\zeta_0)\to w_m(\zeta_0)\,.$$

The second integral converges to $\int_{C_m} P(\zeta_0, d\zeta, D_m-F)w(\zeta)$, where $P(\zeta_0, E, D_m-F)$ $\big(E$ is an elementary set on the boundary $Bd(D_m)\equiv C_m$ of $D_m\big)$ is the probability that the Brownian motion starting from ζ will enter into E for some $t>0$ without entering into $Bd(D_m-F)-E$ $=F\cup(C_m-E)$ before it. This follows simply from the fact that $P(\zeta_0, E, D_{m,n})\to P(\zeta_0, E, D_m-F)$ as $n\to\infty$. Thus we see

$$(23) \qquad w(\zeta_0)=w_m(\zeta_0)+\int_{C_m} P(\zeta_0, d\zeta, D_m-F)w(\zeta)\,.$$

Let us put

$$(24) \qquad \delta=\min_{\zeta\in C_1} w(\zeta)\,.$$

Since F is of positive capacity, so $w(\zeta)\geqq w_2(\zeta)>0$ in D_2-F, and hence we must have $\delta>0$. Since $w(\zeta)$ is bounded and harmonic in R^2-F, so we see that $w(\zeta)\geqq\delta>0$ on R^2-D_1. Consequently, (23) implies

$$(25) \qquad w(\zeta_0)-w_m(\zeta_0)\geqq \delta P(\zeta_0, C_m, D_m-F)$$
$$=\delta\big(1-w_m(\zeta_0)\big)\geqq\delta\big(1-w(\zeta_0)\big)\,.$$

From this follows immediately that $w(\zeta_0)=1$; for, $w(\zeta_0)<1$ would imply $w(\zeta_0)-w_m(\zeta_0)\geqq\delta\big(1-w(\zeta_0)\big)>0$ for $m=1, 2, \dots$, which is clearly a contradiction. Since ζ_0 is an arbitrary point of R^2-F, this completes the proof of Theorem 5.

8. We shall state some theorems which follow easily from Theorem 6.

Theorem 7. Let F be a compact set in R^2 of positive capacity. Then, for any $\zeta \in R^2$, the probability that the Brownian motion starting from ζ will enter into F infinitely many times for infinitely large t is 1.

This is an immediate consequence of the following fact, which follows easily from Theorem 6: for any $\zeta \in R^2$ and for any $t_0 \geqq 0$, the probability that the Brownian motion starting from ζ will enter into F for some t with $t \geqq t_0$ is 1.

Theorem 8. In the Brownian motion in R^2, almost all paths constitute an everywhere dense set in R^2, and come back to any neighborhood of any given point infinitely many times for infinitely large t.[*]

This follows from Theorem 7 and the fact that there exists in the Gaussian plane a sequence of circular domains $\{K_n \mid n=1, 2 \ldots\}$ with the property that for any open set G of R^2 there exists an n such that $K_n \subset G$.