Ordering among the topologies induced by various polynomial norms*

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Abstract

Let us consider the following two norms in the vector space $\mathcal{P}$ of all complex polynomials:

$$\|p\|_{D_r} := \sup\{|p(z)| : |z| < r\}, \text{ and } \|p\|_1 := \sum_{i=0}^{n} |a_i|,$$

where $p(z) = \sum_{i=0}^{n} a_i z^i$. In this note we show that, if $0 < \varepsilon < \varepsilon' < 1 < r < r'$, then

$$\|\cdot\|_{D_\varepsilon} \prec \|\cdot\|_{D_{\varepsilon'}} \prec \|\cdot\|_{D_1} \prec \|\cdot\|_1 \prec \|\cdot\|_{D_r} \prec \|\cdot\|_{D_{r'}},$$

where $\prec$ represents the natural (strict) partial order in their corresponding induced topologies.

1 Introduction and preliminaries

Let us denote by $\mathcal{P}$ and $\mathcal{P}_n$, respectively, the vector spaces of all complex polynomials and all complex polynomials of degree at most $n \in \mathbb{N}$. Since $\mathcal{P}_n$ is finite dimensional, all norms defined on $\mathcal{P}_n$ are equivalent. In other words, if $\|\cdot\|_a$

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and $\|\cdot\|_b$ are two norms defined on $\mathcal{P}_n$, then there exist constants $k(n), K(n) > 0$ such that

$$k(n)\|p\|_a \leq \|p\|_b \leq K(n)\|p\|_a \tag{1.1}$$

for all $p \in \mathcal{P}_n$. Inequalities of this type have been studied in the past for several polynomial norms. For instance, we can endow $\mathcal{P}$ with the following norms:

1. $\|p\|_{D_r} := \sup\{|p(z)| : |z| < r\}$, and
2. $\|p\|_1 := \sum_{i=0}^{n} |a_i|$,

where p is given by $p(z) = \sum_{i=0}^{n} a_i z^i$, $a_0, \ldots, a_n \in \mathbb{C}$, $r > 0$, and $D_r = r\mathbb{D}$ with $\mathbb{D}$ being the open unit disk. The optimal constants $k(n,r), K(n,r') > 0$ in (1.1), where $r, r' > 0$, $\|\cdot\|_a = \|\cdot\|_{D_r}$ and $\|\cdot\|_b = \|\cdot\|_{D_{r'}}$ are known (see for instance [6] and [13] for a complete account on polynomials and polynomial inequalities). A natural question would be whether or not $\|\cdot\|_{D_r}$ and $\|\cdot\|_{D_{r'}}$ are equivalent too in $\mathcal{P}$. The answer is no. However we can establish a relationship between the topologies induced by $\|\cdot\|_{D_r}$ and $\|\cdot\|_{D_{r'}}$ in $\mathcal{P}$.

Given two norms $\|\cdot\|$ and $\|\cdot\|'$ on a vector space Z, we can define a relation representing the natural partial order $(\preceq)$ in their respective induced topologies $T_{\|\cdot\|}$ and $T_{\|\cdot\|'}$ as follows.

Definition 1.1. *We say that $\|\cdot\| \preceq \|\cdot\|'$ if the following three equivalent statements hold:*

(a) *There exists a constant $K > 0$ such that, for all $p \in Z$, we have $\|p\| \leq K\|p\|'$.*

(b) *The identity operator $I : (Z, \|\cdot\|') \to (Z, \|\cdot\|)$ is continuous.*

(c) *$T_{\|\cdot\|'}$ is finer than $T_{\|\cdot\|}$, that is, $T_{\|\cdot\|} \subset T_{\|\cdot\|'}$.*

Remark 1.2. The relation $\preceq$ is *not* really a partial order *on* the collection of all norms on Z, since we can have two equivalent norms which are not equal. Then we should always see $\preceq$ as the natural partial order *on the collection of their induced topologies*, that is,

$$\|\cdot\| \preceq \|\cdot\|' \text{ if and only if } T_{\|\cdot\|} \subset T_{\|\cdot\|'}.$$

We also consider the corresponding strict order relation, that is:

Definition 1.3. *Given two norms $\|\cdot\|$ and $\|\cdot\|'$ on a vector space Z, we say that $\|\cdot\| \prec \|\cdot\|'$ if $\|\cdot\| \preceq \|\cdot\|'$ but $\|\cdot\|' \npreceq \|\cdot\|$.*

The content of the next proposition is well known.

Proposition 1.4. *Let $\|\cdot\|$ and $\|\cdot\|'$ be two norms on a vector space Z. The following are equivalent:*

1. *$\|\cdot\| \prec \|\cdot\|'$.*
2. *The identity operator $I : (Z, \|\cdot\|') \to (Z, \|\cdot\|)$ is continuous but it is not a topological isomorphism.*

A very simple way of proving that $\|\cdot\| \prec \|\cdot\|'$ is by means of compact operators. By B_X we will denote the closed unit ball of a normed space X.

Lemma 1.5. *Let $(E, \|\cdot\|')$, $(F, \|\cdot\|)$ be normed spaces and $Z \subset E, F$ be an infinite dimensional vector space. Suppose that $T : (E, \|\cdot\|') \to (F, \|\cdot\|)$ is a linear operator with $T|_Z = I$, the identity operator. If T is a compact operator then $\|\cdot\| \prec \|\cdot\|'$ on Z.*

Proof. Since T is compact, T is continuous and, thus, the operator $I = T|_Z : (Z, \|\cdot\|') \to (Z, \|\cdot\|)$ is also continuous. Moreover, by Riesz's Theorem (see, e.g., [9]) there exist $\varepsilon_0 > 0$ and a sequence $\{p_n\}_{n\in\mathbb{N}} \subset B_{(Z,\|\cdot\|')}$ with $\|p_n - p_m\|' \geq \varepsilon_0 > 0$. Since T is a compact operator, we can assume, passing to a subsequence if necessary, that $\{T(p_n) = p_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(Z, \|\cdot\|)$. Therefore the operator $I : (Z, \|\cdot\|) \to (Z, \|\cdot\|')$ does not transform Cauchy sequences into Cauchy sequences and it cannot be uniformly continuous (nor continuous, by the linearity of I). Hence its inverse $I : (Z, \|\cdot\|') \to (Z, \|\cdot\|)$ is not a topological isomorphism. By the previous proposition, we conclude that $\|\cdot\| \prec \|\cdot\|'$ on Z. ∎

On the one hand, it follows from the triangle inequality that $\|p\|_{D_1} \leq \|p\|_1$ for all $p \in \mathcal{P}$ (and, thus, $\|\cdot\|_{D_1} \preceq \|\cdot\|_1$). On the other hand, we shall prove that, for all $r > 1$, there exists a constant $K(r) > 0$ such that $\|p\|_1 \leq K(r)\|p\|_{D_r}$ for $p \in \mathcal{P}$ (and, thus, $\|\cdot\|_{D_1} \preceq \|\cdot\|_1 \preceq \|\cdot\|_{D_r}$ for all $r > 1$).

It might seem intuitive the fact that, if $r \to 1^+$, then

$$\|\cdot\|_{D_1} \preceq \|\cdot\|_1 \preceq \|\cdot\|_{D_r} \longrightarrow \|\cdot\|_{D_1}$$

and that, as a consequence, the norms $\|\cdot\|_{D_1}$ and $\|\cdot\|_1$ are really equivalent. However, and as we will also prove, this is *not* true. We shall prove that, although none of these previous norms are actually equivalent in any sense, what we do have is that

$$\|\cdot\|_{D_\varepsilon} \prec \|\cdot\|_{D_{\varepsilon'}} \prec \|\cdot\|_{D_1} \prec \|\cdot\|_1 \prec \|\cdot\|_{D_r} \prec \|\cdot\|_{D_{r'}}$$

for every $0 < \varepsilon < \varepsilon' < 1 < r < r'$. Moreover, it is provided a rather general criterion about topological largeness of sets arising naturally when comparing two norms. The notation will be rather usual and the tools we employ are classical ones from the fields of Topology and Complex Variables.

2 The results

First of all, we shall need some additional notation.

Definition 2.1. *For every $r > 0$, we denote $\mathcal{H}_b(D_r) := \{f \in \mathcal{H}(D_r) : \|f\|_{D_r} < +\infty\}$, where $\mathcal{H}(D_r)$ stands for the space of all holomorphic functions on D_r.*

Remark 2.2. We consider $\mathcal{H}_b(D_r)$ as the Banach space $(\mathcal{H}_b(D_r), \|\cdot\|_{D_r})$ and, naturally, if $0 < r < r'$ then, by the Identity Principle, we may consider $\mathcal{H}_b(D_{r'})$ as a subset of $\mathcal{H}_b(D_r)$.

Definition 2.3. *If $r' > r > 0$, we define the linear operator $I_{r,r'} : \mathcal{H}_b(D_{r'}) \to \mathcal{H}_b(D_r)$ as $I_{r,r'}(f) = f$.*

Now, we can obtain the first of our main results.

Theorem 2.4. *Assume that $0 < r < r'$. Then the following holds:*

1. *The ball $B_{\mathcal{H}_b(D_{r'})}$ is compact in $\mathcal{H}_b(D_r)$.*
2. *$I_{r,r'}$ is a compact operator.*
3. *$\|\cdot\|_{D_r} \prec \|\cdot\|_{D_{r'}}$ on $\mathcal{P}$.*

Proof. Obviously (1) implies (2) and, by Lemma 1.5 (with $E = \mathcal{H}_b(D_{r'})$, $F = \mathcal{H}_b(D_r)$, $Z = \mathcal{P}$, $\|\cdot\| = \|\cdot\|_{D_r}$, and $\|\cdot\|' = \|\cdot\|_{D_{r'}}$), (2) implies (3). So we only have to prove (1).

With this aim, let $\{f_n\}_{n\in\mathbb{N}} \subset B_{\mathcal{H}_b(D_{r'})}$. By Montel's Theorem (see, e.g., [10]) there exist a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ and an $f \in \mathcal{H}(D_{r'})$ such that $f_{n_k} \longrightarrow f$ uniformly on compact sets in $D_{r'}$. We conclude that $f \in B_{\mathcal{H}_b(D_{r'})}$ and $f_{n_k} \longrightarrow f$ in $\mathcal{H}_b(D_r)$. So $B_{\mathcal{H}_b(D_{r'})}$ is compact in $\mathcal{H}_b(D_r)$. ∎

On the other hand, if we now consider $r > 1$, we have that, for all $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}_b(D_r)$, its radius of convergence is not less than $r > 1$, so $\sum_{n=0}^{\infty} |a_n| < +\infty$. This allows us to consider the linear operator given in the next definition, where ℓ_1 denotes the set of all absolutely summable sequences of complex numbers, which becomes a Banach space when endowed with the norm $\|(a_n)_{n\geq 0}\| = \sum_{n=0}^{\infty} |a_n|$.

Definition 2.5. *For all $r > 1$ we define the operator $I_r : \mathcal{H}_b(D_r) \to \ell_1$ as $I_r(f) := (a_n)_{n\geq 0}$, where f is as above.*

In order to prove that I_r is continuous, it will be useful to recall some basic concepts and results related to the compact-open topology.

Definition 2.6. *Let $r > 0$, $f \in \mathcal{H}(D_r)$, $K \subset D_r$ be a compact subset and $\varepsilon > 0$. We define $B_f(K, \varepsilon) := \{g \in \mathcal{H}(D_r) : \sup\{|g(z) - f(z)| : z \in K\} \leq \varepsilon\}$.*

Theorem 2.7. *Let T_c be the compact-open topology in $\mathcal{H}(D_r)$. Then we have:*

1. *The family $\{B_f(K, \varepsilon) : f \in \mathcal{H}(D_r),\ \varepsilon > 0,\ K \text{ compact} \subset D_r\}$ is a neighborhood base for $(\mathcal{H}(D_r), T_c)$.*
2. *$f_n \longrightarrow f$ in $(\mathcal{H}(D_r), T_c)$ if and only if $f_n \longrightarrow f$ uniformly on compact subsets of D_r.*
3. *$(\mathcal{H}(D_r), T_c)$ is a completely metrizable space, hence a Baire space.*

Definition 2.8. *Let $r > 1$. For every $N \in \mathbb{N}$, we denote*

$$F_N := \left\{f \in \mathcal{H}(D_r) : \sum_{n=0}^{\infty} |a_n| \leq N \text{ with } f(z) = \sum_{n=0}^{\infty} a_n z^n\right\}.$$

Remark 2.9. Notice that $\mathcal{H}(D_r) = \bigcup_{N\in\mathbb{N}} F_N$.

The easy proof of the following result is left to the reader. If α is a scalar and S is a subset of a vector space, then αS stands for $\{\alpha x : x \in S\}$.

Lemma 2.10. *Assume that $r > 1$ and $R > 0$. Let $\{(a_{i,n})_{i\geq 0}\}_{n\in\mathbb{N}} \subset RB_{\ell_1}$ be a sequence such that*

$$\lim_{n\to\infty} a_{i,n} = a_i$$

for all $i \in \mathbb{N}$. Then $\sum_{i=0}^{\infty} |a_i| \leq R$.

In the following theorem, we collect a number of properties of the sets F_N given in Definition 2.8.

Theorem 2.11. *Let $r > 1$. We have:*

(a) *The set F_N is a closed subset of $(\mathcal{H}(D_r), T_c)$ for all $N \in \mathbb{N}$.*

(b) *There exists an $N \in \mathbb{N}$ such that:*

  (1) *F_N has non-empty interior in $(\mathcal{H}(D_r), T_c)$.*
  (2) *$0 \in int_{(\mathcal{H}(D_r),T_c)} F_N$.*
  (3) *There exists $\varepsilon > 0$ such that $\varepsilon B_{\mathcal{H}_b(D_r)} \subset F_N$.*

(c) *The operator I_r given in Definition 2.5 is continuous.*

Proof. (a) Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in F_N such that $\lim_{n\to\infty} f_n = f$ in $(\mathcal{H}(D_r), T_c)$, where $f_n(z) = \sum_{i=0}^{\infty} a_{i,n} z^i$ and $f(z) = \sum_{i=0}^{\infty} a_i z^i$. Since $f_n \longrightarrow f$ uniformly on compact sets in D_r, by Weierstrass' Convergence Theorem (see [1, pp. 176–177]) we get $\lim_{i\to\infty} f_n^{(i)}(0) = f^{(i)}(0)$ for all $i \in \mathbb{N} \cup \{0\}$, and hence $\lim_{i\to\infty} a_{i,n} = a_i$. By Lemma 2.10, $\sum_{i=0}^{\infty} |a_i| \leq N$ and so $f \in F_N$.
(b) Part (1) follows from (a), Theorem 2.7(3) and Remark 2.9.
(2) By (1) and Theorem 2.7(1), there exist $f \in \mathcal{H}(D_r)$, a compact set $K \subset D_r$ and $\varepsilon > 0$ such that $f + B_0(K,\varepsilon) = B_f(K,\varepsilon) \subset F_N$. In particular, $f \in F_N$. Now, observe that F_N is absolutely convex, that is, $\alpha g + \beta h \in F_N$ for all $g, h \in F_N$ and all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| + |\beta| \leq 1$. Consequently, $-\frac{1}{2} f + \frac{1}{2} B_f(K,\varepsilon) \subset F_N$. But $-\frac{1}{2} f + \frac{1}{2} B_f(K,\varepsilon) = B_0(K, \frac{\varepsilon}{2})$, which proves that 0 is in the interior of F_N.
(3) Obviously $\varepsilon B_{\mathcal{H}_b(D_r)} \subset B_0(K,\varepsilon) \subset F_N$.
(c) From (3) above as applied to $N = 1$ we obtain the existence of an $\varepsilon > 0$ such that $\varepsilon B_{\mathcal{H}_b(D_r)} \subset F_1$. But this together with the homogeneity of norms tells us that $\|I_r f\| \leq \frac{1}{\varepsilon} \|f\|_r$ for all $f \in \mathcal{H}_b(D_r)$, which yields the desired continuity. ■

Now, one of our main results can be easily derived:

Theorem 2.12. *Assume that $r > 1$. Then the following holds:*

(a) *The linear operator $I_r : \mathcal{H}_b(D_r) \to \ell_1$ is compact.*

(b) $\|\cdot\|_1 \prec \|\cdot\|_{D_r}$ *on $\mathcal{P}$.*

Proof. By Theorem 2.11(b)(3), I_r is a bounded operator. Fix any $d \in (1, r)$. We can see I_r as the composition $I_r = I_d I_{d,r}$. Since I_d is continuous and $I_{d,r}$ is compact, I_r is compact. This proves (a). Finally, (b) follows from (a) and Lemma 1.5. ■

Given a normed space E, we shall denote by $\overline{E}$ its completion.

Remark 2.13. Let us recall that:

1. $\overline{(\mathcal{P}, \|\cdot\|_1)} = \ell_1$.
2. $\overline{(\mathcal{P}, \|\cdot\|_{D_1})} = \mathcal{H}(D_1) \cap \mathcal{C}(\overline{D_1}) =: \mathcal{A}(D_1)$, the disk algebra.

The content of the following auxiliary assertion is well known.

Lemma 2.14. *Let E and F be normed spaces and let $T : E \to F$ be a linear and continuous operator. Then the following holds:*

1. *There exists a unique linear continuous operator $\overline{T} : \overline{E} \to \overline{F}$ such that $\overline{T}|_E = T$.*
2. *If T is a topological isomorphism then $\overline{T}$ is also a topological isomorphism.*

We denote by $I : \mathcal{P} \to \mathcal{P}$ the identity mapping $I(P) = P$, where the space $\mathcal{P}$ on the left should be thought as identified with c_{00}, the space of eventually zero complex sequences.

Corollary 2.15. *The linear operator $\overline{I} : (a_n)_{n\in\mathbb{N}} \in \ell_1 \mapsto f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{A}(D_1)$ is continuous and injective.*

Proof. By using the Weierstrass M-test, the series $\sum_{n=0}^{\infty} a_n z^n$ converges uniformly on $\overline{D_1}$. Since each term $a_n z^n$ is continuous on $\overline{D_1}$, so is the sum f. Moreover, the Weierstrass convergence theorem guarantees that f is holomorphic in D_1, so that $f \in \mathcal{A}(D_1)$ and the mapping $(a_n)_{n\in\mathbb{N}} \in \ell_1 \mapsto f \in \mathcal{A}(D_1)$ is well defined and, obviously, linear. That this mapping equals $\overline{I}$ is clear because its restriction to $\mathcal{P}$ equals I (via the identification $\mathcal{P} = c_{00}$), and $\mathcal{P}$ is dense both in ℓ_1 and $\mathcal{A}(D_1)$. The continuity of $\overline{I}$ is derived from Lemma 2.14(1), while its injectivity follows from the uniqueness of the Taylor coefficients around 0. ∎

However, is $\overline{I}$ also a topological isomorphism? In order to answer this question, let us focus on the following four conjectures.

Conjectures 2.16.
(CI) $\|\cdot\|_1$ *and* $\|\cdot\|_{D_1}$ *are equivalent norms in* $\mathcal{P}$.
(CII) *The linear operator* $\overline{I} : \ell_1 \to \mathcal{A}(D_1)$ *is a topological isomorphism.*
(CIII) *For every* $f \in \mathcal{A}(D_1)$ *there exists* $(a_i)_{i\in\mathbb{N}} \in \ell_1$ *such that* $f(z) = \sum_{i=0}^{\infty} a_i z^i$ *for all* $z \in \overline{D_1}$.
(CIV) *The set*

$$A := \left\{ f \in \mathcal{A}(D_1) : \sum_{i=0}^{\infty} |a_i| < +\infty \text{ where } f(z) = \sum_{i=0}^{\infty} a_i z^i \ \ \forall z \in D_1 \right\} \tag{2.1}$$

is of second category in $\mathcal{A}(D_1)$.

Proposition 2.17. *The previous four conjectures* (CI), (CII), (CIII) *and* (CIV) *are equivalent.*

Proof. To start with, the facts (CII) $\Longrightarrow$ (CIII) and (CIII) $\Longrightarrow$ (CIV) are straightforward.

- (CI) is equivalent to (CII): $\|\cdot\|_1$ and $\|\cdot\|_{D_1}$ are equivalent norms in $\mathcal{P}$ if and only if $I : (\mathcal{P}, \|\cdot\|_1) \to (\mathcal{P}, \|\cdot\|_{D_1})$ is a topological isomorphism and, by Remark 2.13 and Lemma 2.14, the last property is equivalent to the fact that $\overline{I} : \ell_1 \to \mathcal{A}(D_1)$ is a topological isomorphism.
- (CIV) $\Longrightarrow$ (CII): The set $\overline{I}(\ell_1) = A$ is a second category set. Now, since $\overline{I}$ is linear, continuous and injective, the Banach–Schauder Theorem (Open Mapping Theorem) implies that $\overline{I}$ is a topological isomorphism. ∎

Proposition 2.18. (CIII) *is false, and so are* (CI), (CII) *and* (CIV) *by Proposition 2.17.*

Proof. The following result can be found in [8, p. 77]: There exists $f \in \mathcal{A}(D_1)$ such that $\sum_{i=0}^{\infty} |a_i| = +\infty$, where $f(z) = \sum_{i=0}^{\infty} a_i z^i$ for all $z \in D_1$. This disproves (CIII). ∎

Since (CI) is false but we have $\|\cdot\|_{D_1} \preceq \|\cdot\|_1$, we obtain another promised result:

Theorem 2.19. $\|\cdot\|_{D_1} \prec \|\cdot\|_1$ *on* $\mathcal{P}$.

We also obtain the following consequence.

Corollary 2.20. *Consider the operators* I_r $(r > 1)$ *given in Definition 2.5. Then* $\|I_r\| \longrightarrow +\infty$ *as* $r \to 1^+$.

Proof. By way of contradiction, suppose that there exist $K > 0$ and a sequence $\{r_n\}_{n\in\mathbb{N}}$ with $r_n \to 1^+$ such that $\|I_{r_n}\| \le K$ for all $n \in \mathbb{N}$. We have that, for every $n \in \mathbb{N}$ and for all $p \in \mathcal{P}$,

$$\|p\|_1 = \|I_{r_n}(p)\|_1 \le K\|p\|_{D_{r_n}} \longrightarrow K\|p\|_{D_1} \text{ as } n \to \infty.$$

Thus, $\|p\|_1 \le K\|p\|_{D_1}$ and so $\|\cdot\|_1 \preceq \|\cdot\|_{D_1}$, which is absurd. ∎

Let A be the set defined in (2.1). Since (CIV) is false, A is a first category set. We will show that A enjoys, actually, a nice topological structure; namely, A is an $\mathcal{F}_\sigma$ set. Note that, in addition, A is dense since it contains the class $\mathcal{P}$.

Let $f \in \mathcal{A}(D_1)$ with $f(z) = \sum_{i=0}^{\infty} a_i z^i$ for all $z \in D_1$. We know that its radius of convergence is at least 1. Then, for all $\varepsilon \in (0,1)$, we obtain $\sum_{i=0}^{\infty} |a_i|\varepsilon^i < +\infty$. Thus, we can define the following operator.

Definition 2.21. *For every* $\varepsilon \in (0,1)$ *we define the linear operator*

$$i_\varepsilon : \mathcal{A}(D_1) \longrightarrow \ell_1$$

as

$$i_\varepsilon(f) = (a_i \varepsilon^i)_{i\ge 0},$$

where $f(z) = \sum_{i=0}^{\infty} a_i z^i$ *for every* $z \in D_1$.

Proposition 2.22. *For every* $\varepsilon \in (0,1)$*, we have that* i_ε *is a compact operator.*

Proof. We are going to define a linear operator

$$T_\varepsilon : \mathcal{A}(D_1) \to \mathcal{H}_b(D_{1/\varepsilon}).$$

For this, making the substitution $z = \varepsilon\omega$ we set $T_\varepsilon(f)(\omega) := f(\varepsilon\omega)$. Now, since

$$\|T_\varepsilon(f)\|_{D_{1/\varepsilon}} = \|f\|_{D_1}$$

for every $f \in \mathcal{A}(D_1)$, we have that T_ε is continuous. Moreover, we can see i_ε as the composition $i_\varepsilon = I_{1/\varepsilon}T_\varepsilon$, where $I_{1/\varepsilon}$ is a compact operator. To sum up, i_ε is the composition of a compact operator and of a continuous operator, from which we conclude that it is compact. ∎

Corollary 2.23. *For every* $\varepsilon \in (0,1)$ *and every* $M > 0$*, we have that*

$$C_{M,\varepsilon} := \left\{ f \in \mathcal{A}(D_1) : \sum_{i=0}^{\infty} |a_i|\varepsilon^i \leq M \textit{ where } f(z) = \sum_{i=0}^{\infty} a_i z^i \ \ \forall z \in D_1 \right\}$$

is a closed subset of $\mathcal{A}(D_1)$*.*

Proof. It suffices to notice that $C_{M,\varepsilon} = i_\varepsilon^{-1}(MB_{\ell_1})$. ∎

Corollary 2.24. *The set*

$$C_M := \left\{ f \in \mathcal{A}(D_1) : \sum_{i=0}^{\infty} |a_i| \leq M \textit{ where } f(z) = \sum_{i=0}^{\infty} a_i z^i \ \ \forall z \in D_1 \right\} \tag{2.2}$$

is closed for every $M > 0$*.*

Proof. Let us show that $C_M = \bigcap_{\varepsilon\in(0,1)} C_{M,\varepsilon}$. It is clear that $C_M \subset \bigcap_{\varepsilon\in(0,1)} C_{M,\varepsilon}$. Let us see that $C_M \supset \bigcap_{\varepsilon\in(0,1)} C_{M,\varepsilon}$. If $f \in \bigcap_{\varepsilon\in(0,1)} C_{M,\varepsilon}$ (where $f(z) = \sum_{i=0}^{\infty} a_i z^i$ for all $z \in D_1$) then (taking a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ with $\varepsilon_n \to 1^-$) we have that, for every $n \in \mathbb{N}$, $\sum_{i=0}^{\infty} |a_i|\varepsilon_n^i \leq M$. Finally, by Lemma 2.10, $\sum_{i=0}^{\infty} |a_i| \leq M$ and $f \in C_M$, which concludes the proof. ∎

Theorem 2.25. *The set* A *defined in* (2.1) *is an* $\mathcal{F}_\sigma$ *set of first category in* $\mathcal{A}(D_1)$*. Hence the set*

$$\left\{ f \in \mathcal{A}(D_1) : \sum_{i=0}^{\infty} |a_i| = +\infty \textit{ where } f(z) = \sum_{i=0}^{\infty} a_i z^i \ \ \forall z \in D_1 \right\}$$

is a dense G_δ *set, so residual in* $\mathcal{A}(D_1)$*.*

Proof. It suffices to notice that A is of first category due to Proposition 2.18 and to the fact that we can write $A = \bigcup_{N\in\mathbb{N}} C_N$, where the C_N's are given in (2.2). ∎

3 Topological and algebraic genericity

To finish this paper, and inspired by the last theorem and its proof, we can furnish a rather general criterion of topological largeness inside normed spaces, see Theorem 3.3 below. Actually, the criterion also contains assertions about algebraic largeness.

Let us denote by $[-\infty,+\infty]$ the extended real line, endowed with the order topology. Recall that, if X is a topological space, a mapping $\Phi : X \to [-\infty,+\infty]$ is called *lower semicontinuous* (see, e.g., [7] for concepts and properties) whenever, given any $\alpha \in \mathbb{R}$, the set $\{x \in X : \Phi(x) > \alpha\}$ is open. If X is a metric space, this is equivalent to $\Phi(x_0) \le \liminf_{x \to x_0} \Phi(x)$ for all $x_0 \in X$.

In the next definition, we are considering on $[0,+\infty]$ the natural extension of the usual order in $[0,+\infty)$.

Definition 3.1. *Let X be a vector space and $\Phi : X \to [0,+\infty]$ be a lower semicontinuous mapping. We say that Φ is an* extended norm on X *provided that the following properties are satisfied:*

(i) $\Phi(x) = 0$ *if and only if* $x = 0$.

(ii) *If* $\{\Phi(x),\Phi(y)\} \subset [0,+\infty)$ *then* $\Phi(x+y) \le \Phi(x) + \Phi(y)$.

(iii) *If* $\Phi(x) < +\infty$ *and* α *is a scalar then* $\Phi(\alpha\, x) = |\alpha|\Phi(x)$.

It is easy to see that, under the notation of the last definition, the set

$$X_\Phi := \{x \in X : \Phi(x) < +\infty\}$$

is a vector subspace of X and that the restriction of Φ to X_Φ is a norm on X_Φ.

The following concepts, which are taken from the theory of lineability (see [2, 3, 5, 12, 14] for the necessary background and early results within this theory) are also needed.

Definition 3.2. *Assume that X is a vector space and that $A \subset X$. We say that A is* lineable *if it contains, except for zero, an infinite dimensional vector space. If X is, in addition, a topological vector space, then A is said to be* dense-lineable (spaceable, resp.) *in X provided that it contains, except for zero, a dense (a closed infinite dimensional, resp.) vector subspace.*

Theorem 3.3. *Assume that $(X, \|\cdot\|)$ is a Banach space and that Φ is an extended norm on X. Let us denote*

$$A_\infty := X \setminus X_\Phi = \{x \in X : \Phi(x) = +\infty\}.$$

Then the following holds:

(a) *If $\Phi \not\preceq \|\cdot\|$ on X_Φ, then the set A_∞ is residual in X.*

(b) *If X_Φ is dense in X, $\|\cdot\| \prec \Phi$ on X_Φ and (X_Φ, Φ) is a Banach space, then A_∞ is spaceable in X. If, in addition, $(X, \|\cdot\|)$ is separable, then A_∞ is dense-lineable too.*

Proof. (a) We have to prove that X_Φ is of first category in X. For this, note that $X_\phi = \bigcup_{n=1}^\infty F_n$, where we have set $F_\alpha := \{x \in X : \Phi(x) \le \alpha\}$ for every ,$\alpha \in (0, +\infty)$. Therefore it suffices to show that each set F_α is closed and has empty interior in X. That F_α is closed is derived from the openness of $X \setminus F_\alpha = \{x \in X : \Phi(x) > \alpha\}$, which in turn comes from the assumption of lower semicontinuity for Φ.

In order to prove that F_α has empty interior in X, assume, by way of contradiction, that there are $x_0 \in X$ and $R > 0$ such that $\{x \in X : \|x - x_0\| \le R\} \subset F_\alpha$ or, equivalently, $\Phi(x) \le \alpha$ for every $x \in X$ satisfying $\|x - x_0\| \le R$. Since $\Phi \not\preceq \|\cdot\|$ on X_Φ, we can find a sequence $\{x_n\}_{n\ge1} \subset X_\Phi$ such that $\Phi(x_n) > n\|x_n\|$ for all $n \in \mathbb{N}$. Note that $x_n \neq 0$, and so $\|x_n\| > 0$ $(n = 1, 2, \dots)$. Select an $N \in \mathbb{N}$ with $N > 2\alpha/R$ and define $x := x_0 + \frac{R}{\|x_N\|} x_N$. Note, on the one hand, that $\|x - x_0\| = R$, which implies $\Phi(x) \le \alpha$. But, on the other hand, since Φ is a norm on X_Φ, by the triangle inequality we obtain

$$\Phi(x) \ge \Phi\Big(\frac{R}{\|x_N\|} x_N\Big) - \Phi(x_0) > R\,N - \alpha > 2\,\alpha - \alpha = \alpha,$$

which is absurd. This proves the residuality of A_∞.

(b) Here we shall make use of the following facts. The first of them is a special case of Theorem 3.3 in [11], while the second one can be found in [4, Theorem 2.5]:

(1) *Let Y be a Banach space and X be a Fréchet space. If $T : Y \to X$ is a continuous linear mapping and $T(Y)$ is not closed in X then the complement $X \setminus T(Y)$ is spaceable in X.*

(2) *Let X be a metrizable separable topological vector space and Y be a vector subspace of X. If $X \setminus Y$ is lineable then $X \setminus Y$ is dense-lineable in X.*

Let us apply (1) with $Y := (X_\Phi, \Phi)$ and $T := I : x \in X_\Phi \mapsto x \in X$, the inclusion mapping, which is linear, but also continuous because $\|\cdot\| \preceq \Phi$ on X_Φ. Observe that, under this notation, $A_\infty = X \setminus T(Y)$. Assume, via contradiction, that $T(Y) = X_\Phi$ is closed in X. Since $\|\cdot\| \prec \Phi$, we have in particular that $\Phi \not\preceq \|\cdot\|$ on X_Φ. Hence, by part (a), A_∞ is residual in X, so nonempty. But $X_\Phi = X$ because X_Φ is dense and closed, which entails $A_\infty = \varnothing$, that is absurd. Consequently, $T(Y)$ is not closed in X and (1) tells us that A_∞ is spaceable.

Finally, if we assume that $(X, \|\cdot\|)$ is separable, the dense-lineability of A_∞ follows from the above result (2) (with $Y := X_\Phi$) and the fact that spaceability implies lineability. ∎

Remark 3.4. Theorem 2.25 follows from Theorem 3.3(a) just by taking $X := \mathcal{A}(D_1)$, $\|f\| := \sup_{z \in D_1} |f(z)|$ and $\Phi(f) := \sum_{n=0}^\infty |a_n|$, where $f \in \mathcal{A}(D_1)$ and $f(z) = \sum_{n=0}^\infty a_n z^n$ for all $z \in D_1$. Since $\|\cdot\| \prec \Phi$ on the space $A := \big\{f \in \mathcal{A}(D_1) : \sum_{n=0}^\infty |a_n| < +\infty\big\}$, we obtain in particular that $\Phi \not\preceq \|\cdot\|$. Then the unique property to be checked is the lower semicontinuity of Φ. For this, observe that each mapping

$$S_n : f \in \mathcal{A}(D_1) \longmapsto \sum_{k=0}^n |a_k| = \sum_{k=0}^n \Big|\frac{f^{(k)}(0)}{k!}\Big| \in \mathbb{R} \quad (n \in \mathbb{N})$$

is continuous, due to the Weierstrass convergence theorem for derivatives and the fact that convergence in $\mathcal{A}(D_1)$ implies uniform convergence on compacta (hence convergence at 0). In particular, each S_n is lower semicontinuous. But, evidently, $\Phi = \sup\{S_n : n \in \mathbb{N}\}$, and the supremum of a family of lower semicontinuous functions is known to be lower semicontinuous (see [7]). The spaceability (already proved in [11]) and the dense lineability of $\{f \in \mathcal{A}(D_1) : \sum_{n=0}^{\infty} |a_n| = +\infty\}$ follow from Theorem 3.3(b) since the set A is dense in the (separable) space $\mathcal{A}(D_1)$ and (A, Φ) is a Banach space.

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