On a population model with nonlinear boundary conditions arising in ecosystems

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Abstract

In this paper we study a model of population which is described by positive solutions to the nonlinear boundary value problem

$$\begin{cases} -\Delta u = au - bu^2 - c\frac{u^2}{1+u^2} - \epsilon, & x \in \Omega, \\ \mathbf{n}.\nabla u + g(u) = 0, & x \in \partial\Omega. \end{cases}$$

Here Ω is a bounded smooth domain of $\mathbb{R}^N$, Δ is the Laplacian operator, a, b, c, ϵ are positive parameters and $g \in C^1\Big([0,\infty),\, [\theta,\infty)\Big)$ is decreasing for some $\theta > 0$. This model describes the dynamics of the fish populations. Our existence results are established via the well-known sub-super solution method.

1 Introduction

In this paper we study the following population model with nonlinear boundary conditions:

$$\begin{cases} -\Delta u = au - bu^2 - c\frac{u^2}{1+u^2} - \epsilon, & x \in \Omega, \\ \mathbf{n}.\nabla u + g(u) = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with sufficiently smooth boundary and $N \geq 1$, Δ is the Laplace operator, a, b, c, ϵ are positive parameters and $g \in C^1([0,\infty), [\theta,\infty))$ is decreasing for some $\theta > 0$.

Received by the editors in August 2016 - In revised form in September 2018.

Communicated by D. Bonheure.

2010 *Mathematics Subject Classification* : 35J55, 35J65.

Key words and phrases : Population model; Nonlinear boundary conditions; Sub-supersolutions.

(1) arises from population biology of one species. Here u is the population density and $aubu^2$ represents logistics growth. This model describes a logistically growing species with grazing of a fixed number of grazers and constant yield harvesting (see [10, 11]). The assumptions are that the ecosystem is spatially homogeneous and the herbivore density is a constant. The rate of grazing is given by $\frac{u^2}{1+u^2}$. In addition, most ecological systems have some form of predation or harvesting of the population. For example, hunting or fishing is often used as an effective means of wildlife management. Here ϵ is the rate of the harvesting distribution. This model has also been applied to describe the dynamics of fish populations (see [13, 14]). In such cases, the term $\frac{u^2}{1+u^2}$ corresponds to natural predation.

In the literature, the homogeneous Dirichlet boundary condition, $u = 0; \partial\Omega$, Neumann boundary condition, $\frac{\partial u}{\partial n}; \partial\Omega$, and linear combinations of the two aforementioned boundary conditions (known as a Robin boundary condition) have been employed almost exclusively in reaction diffusion population models. Use of linear boundary conditions assumes that the behavior of the population on the boundary is independent of the population density itself. However, ecologists have reported density dependent emigration rates from patches of habitat. Empirical studies conducted by several ecologists have even shown a negative correlation between density and emigration rates, in which animals have a tendency to leave a patch when density is low and stay in the patch when it is high. This fact brings into question a commonly made assumption in ecology, that animals unilaterally exhibit positive density dependent dispersal and patch emigration (see [12, 15, 16]).

The motivation for this study cames from the work in [2] where the authors established the existence of positive solutions to such problems with Dirichlet boundary conditions. Hence, the main purpose of this paper is to initiate extension of their results to the nonlinear boundary conditions. Here, $g(u)$ represents the population that remains on the boundary when reached. One can refer to [4, 5, 6, 8, 9] for some recent existence results of population models.

2 Main results

In this section we give our main results. First we prove some nonexistence results.

Theorem 2.1. Let $\eta = \inf_{u\in[0,\infty)} g(u)$. Then there is a constant a^*, such that (1) has no positive solution for $a \leq a^*$.

Proof. Let $\mu_1 > 0$ and $\phi > 0$ be the first eigenvalue and corresponding positive eigenfunction of

$$\begin{cases} -\Delta\phi = \mu_1\phi, & x \in \Omega, \\ \mathbf{n}.\nabla\phi = -\phi, & x \in \partial\Omega. \end{cases} \tag{2}$$

Define $a^* = \min\{\mu_1, \eta b\}$. For $a \leq a^*$, multiplying (1) by ϕ, and integrating over Ω, we obtain

$$\int_\Omega (-\Delta u)\phi dx = \int_\Omega (au\phi - bu^2\phi - c\phi\frac{u^2}{1+u^2} - \epsilon\phi)dx.$$

But by Green's identity we have

$$\begin{aligned}\int_\Omega (-\Delta u)\phi dx &= \int_\Omega (-\Delta\phi)udx + \int_{\partial\Omega}(\frac{\partial\phi}{\partial n}u - \frac{\partial u}{\partial n}\phi)ds \\ &= \int_\Omega \mu_1\phi udx + \int_{\partial\Omega}(\phi g(u) - \phi u)ds.\end{aligned}$$

Thus we have

$$\int_{\partial\Omega}(\phi g(u) - \phi u)ds = \int_\Omega \left[(a-\mu_1)u\phi - bu^2\phi - c\phi\frac{u^2}{1+u^2} - \epsilon\phi\right]dx. \tag{3}$$

Since $a < \mu_1$, we can see that the right-hand side of (3) is negative. By the maximum principle, we know that $\|u\|_\infty \leq \frac{a}{b} < \eta$ which gives

$$\int_{\partial\Omega}(\phi g(u) - \phi u)ds = \int_{\partial\Omega}(g(u) - u)\phi ds > 0,$$

and by this contradiction Theorem 2.1 is proven. ■

Theorem 2.2. Let $a > a^*$, $b > 0$ and $c > 0$ be fixed. If

$$\epsilon > \frac{a(a-\mu_1)}{b},$$

then (1) has no nonnegative solution.

Proof. From (3) we obtain,

$$\epsilon\int_\Omega \phi dx \leq (a-\mu_1)\int_\Omega u\phi dx \leq \frac{a(a-\mu_1)}{b}\int_\Omega \phi dx,$$

a contradiction when (4) holds. ■

Next, we shall establish our existence results via the method of sub and super-solutions. By a sub-solution of (1) we mean a function $\psi : \bar{\Omega} \to \mathbb{R}$ satisfying:

$$\begin{cases} -\Delta\psi \leq a\psi - b\psi^2 - c\frac{\psi^2}{1+\psi^2} - \epsilon, & x \in \Omega, \\ \mathbf{n}.\nabla\psi + g(\psi) \leq 0, & x \in \partial\Omega, \end{cases} \tag{4}$$

and by a super-solution of (1) we mean a function $Z : \bar{\Omega} \to \mathbb{R}$ satisfying:

$$\begin{cases} -\Delta Z \geq aZ - bZ^2 - c\frac{Z^2}{1+Z^2} - \epsilon, & x \in \Omega, \\ \mathbf{n}.\nabla Z + g(Z) \geq 0, & x \in \partial\Omega. \end{cases} \tag{5}$$

By strict sub and super-solutions we understand functions ψ and Z for which strict inequalities (4) and (5) hold.

It is well known that if there exist sub and supersolutions ψ and Z respectively of (1) such that $\psi \leq Z$. Then (1) has a solution u such that $u \in [\psi, Z]$ (see [1, 7]).

Theorem 2.3. Let $b > 0$ and $c > 0$ be fixed. Then there exist positive constant a_* and ϵ^* such that (1) has a positive solution for $a > a_*$ and $\epsilon < \epsilon^*$.

Proof. Let λ_1 is the principle eigenvalue for Laplace's equation with Dirichlet boundary conditions. From an anti-maximum principle (see [3]), there exists a $\sigma > 0$ such that the solution z_λ of

$$\begin{cases} -\Delta z - \lambda z = -1, & x \in \Omega, \\ z = 0, & x \in \partial\Omega. \end{cases} \tag{6}$$

for $\lambda \in I = (\lambda_1, \lambda_1 + \sigma)$, is positive for $x \in \Omega$ and is such that $\frac{\partial z_\lambda}{\partial n} < 0$ for $x \in \partial\Omega$, where n is outward normal vector at Ω. Let $\alpha_\lambda = \|z_\lambda\|_\infty$, $m_\lambda = \inf\{m : \frac{\partial(mz_\lambda)}{\partial n} \leq -g(0) - 1\}$, and $a_* = \inf_{\lambda \in I} \max\{2\lambda, 2(b+c)m_\lambda\alpha_\lambda\}$. For $a > a_*$, we can choose $\lambda^* \in I$ such that $a > \max\{2\lambda^*, 2(b+c)m_{\lambda^*}\alpha_{\lambda^*}\}$. For $\gamma \geq 1$, let

$$K = \frac{(a - \lambda^*)\gamma\alpha_{\lambda^*} + (\gamma - 1)}{(b+c)\gamma^2\alpha_{\lambda^*}^2}.$$

First, we state and prove an important claim:

Claim. If $a > \max\{2\lambda^*, 2(b+c)m_{\lambda^*}\alpha_{\lambda^*}\}$, then

$$\frac{m_{\lambda^*}}{\gamma} < \min\{\epsilon, \frac{a}{2\gamma\alpha_{\lambda^*}(b+c)}\}.$$

To prove the claim, we note that $\frac{m_{\lambda^*}}{\gamma} < \frac{a}{2\gamma\alpha_{\lambda^*}(b+c)}$ follows from $a > 2(b+c)m_{\lambda^*}\alpha_{\lambda^*}$. Now, since $a > 2\lambda^*$, $a > 2(b+c)m_{\lambda^*}\alpha_{\lambda^*}$, and $\gamma \geq 1$, the following are true:

$$\begin{aligned}
&\frac{a}{2} - \lambda^* > 0, \\
&a\alpha_{\lambda^*} - \lambda^*\alpha_{\lambda^*} - \frac{a}{2}\alpha_{\lambda^*} > 0, \\
&a\alpha_{\lambda^*} - \lambda^*\alpha_{\lambda^*} - m_{\lambda^*}(b+c)\alpha_{\lambda^*}^2 + 1 > 1, \\
&(a\alpha_{\lambda^*} - \lambda^*\alpha_{\lambda^*} - m_{\lambda^*}(b+c)\alpha_{\lambda^*}^2 + 1)\gamma > 1, \\
&(a - \lambda^*)\gamma\alpha_{\lambda^*} + \gamma - 1 > m_{\lambda^*}(b+c)\alpha_{\lambda^*}^2\gamma.
\end{aligned}$$

Hence

$$K = \frac{(a - \lambda^*)\gamma\alpha_{\lambda^*} + (\gamma - 1)}{(b+c)\gamma^2\alpha_{\lambda^*}^2} > \frac{m_{\lambda^*}}{\gamma}.$$

which prove the claim.

Next, let $\epsilon^* = \min\{\epsilon, \frac{a}{2\gamma\alpha_{\lambda^*}(b+c)}\}$. Now for $\epsilon < \epsilon^*$ there exists a l_c such that

$$\max\{\epsilon, \frac{m_{\lambda^*}}{\gamma}\} < l_c < \epsilon^*.$$

Define $\psi = \gamma l_c z_{\lambda^*}$. A calculation shows that

$$-\Delta\psi = -\Delta(\gamma l_c z_{\lambda^*}) = \gamma l_c(-\Delta z_{\lambda^*}) = \lambda^*\gamma l_c z_{\lambda^*} - \gamma l_c.$$

Hence, if we prove

$$(a-\lambda^*)\gamma z_{\lambda^*} - (b+c)l_c(\gamma z_{\lambda^*})^2 + (\gamma - 1) \geq 0, \tag{7}$$

then

$$\begin{aligned}
-\Delta\psi &= \lambda^*\gamma l_c z_{\lambda^*} - \gamma l_c \\
&\leq a(\gamma l_c z_{\lambda^*}) - (b+c)(\gamma l_c z_{\lambda^*})^2 - l_c \\
&\leq a(\gamma l_c z_{\lambda^*}) - b(\gamma l_c z_{\lambda^*})^2 - c(\gamma l_c z_{\lambda^*})^2 - \epsilon \\
&\leq a(\gamma l_c z_{\lambda^*}) - b(\gamma l_c z_{\lambda^*})^2 - c(\gamma l_c z_{\lambda^*})^2 + \frac{(\gamma l_c z_{\lambda^*})^4}{1+(\gamma l_c z_{\lambda^*})^2} - \epsilon \\
&= a(\gamma l_c z_{\lambda^*}) - b(\gamma l_c z_{\lambda^*})^2 - c\frac{(\gamma l_c z_{\lambda^*})^2}{1+(\gamma l_c z_{\lambda^*})^2} - \epsilon \\
&= a\psi - b\psi^2 - c\frac{\psi^2}{1+\psi^2} - \epsilon,
\end{aligned}$$

and on $\partial\Omega$,

$$\frac{\partial\psi}{\partial n} = \gamma l_c \frac{\partial z_{\lambda^*}}{\partial n} < m_{\lambda^*}\frac{\partial z_{\lambda^*}}{\partial n} \leq -g(0) - 1.$$

To establish (8) we consider $G(t) = (a-\lambda^*)t - (b+c)l_c t^2 + (\gamma - 1) \geq 0$ for all $t \in [0, \gamma\alpha_{\lambda^*}]$. Notice that $G(0) = \gamma - 1 \geq 0$, $G'(0) = a - \lambda^* > 0$, and $G''(0) = -2(b+c)l_c < 0$. Hence $G(t) \geq 0$ for all $t \in [0, \gamma\alpha_{\lambda^*}]$ if $G(\gamma\alpha_{\lambda^*}) = (a-\lambda^*)t - (b+c)l_c(\gamma\alpha_{\lambda^*})^2 + (\gamma - 1) \geq 0$. This easily follows from the fact that $l_c < K$.

To construct a subsolution $\hat{\psi} > 0$; $\bar{\Omega}$, let $\hat{f}(u) = au - (b+c)u^2 - l_c$. Then $\hat{f}$ is increasing on $[0, \frac{a}{2(b+c)}]$. Since $\gamma l_c \alpha_{\lambda^*} < \frac{a}{2(b+c)}$, there is an $\delta > 0$ such that $\gamma l_c \alpha_{\lambda^*} + \delta < \frac{a}{2(b+c)}$, and $g(\delta) \leq g(0) + 1$.

Now define $\hat{\psi} = \psi + \delta$, then $\|\hat{\psi}\|_\infty = l_c\gamma\alpha_{\lambda^*} + \delta$. Also,

$$\begin{aligned}
-\Delta\hat{\psi} &= -\Delta\psi \leq \hat{f}(\psi) < \hat{f}(\psi+\delta) = \hat{f}(\hat{\psi}) \\
&< a\hat{\psi} - b\hat{\psi}^2 - c\hat{\psi}^2 - c\frac{\hat{\psi}^4}{1+\hat{\psi}^2} - l_c < a\hat{\psi} - b\hat{\psi}^2 - c\frac{\hat{\psi}^2}{1+\hat{\psi}^2} - \epsilon,
\end{aligned}$$

and on $\partial\Omega$,

$$\frac{\partial\hat{\psi}}{\partial n} = \frac{\partial\psi}{\partial n} \leq -g(0) - 1 \leq -g(\delta) < -g(\psi+\delta) = g(\hat{\psi}).$$

Thus $\hat{\psi}$ is a subsolution to (1) and it is easy to see that $\hat{\psi} > 0; \bar{\Omega}$.

Next, we construct a supersolution. Choose a large constant $M > 0$ such that $aM - bM^2 - c\frac{M^2}{1+M^2} - \epsilon \leq 0$ and $M \geq \hat{\psi}$ for $x \in \bar{\Omega}$. Let $Z = M$. Then

$$-\Delta Z = 0 > aZ - bZ^2 - c\frac{Z^2}{1+Z^2} - \epsilon, \quad x \in \Omega,$$

and on $\partial\Omega$

$$n.\nabla Z + g(Z) = n.\nabla(M) + g(M) \geq 0.$$

Thus Z is a positive supersolution of (1) for $a > a_*$ and $\epsilon < \epsilon^*$ satisfying $Z \geq \hat{\psi}$ and Theorem 2.3 is proven. ■

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