Gevrey series in compensators linearizing a planar resonant vector field and its unfolding

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Abstract

We consider a planar vector field X near a saddle type $p : -q$ resonant singular point. Assuming that it has a normal form with a Gevrey-d expansion (like $d = p + q$ which is in particular the case when starting from an analytic vector field) we show that X can be linearized working with a change of coordinates that is of Gevrey order d in certain log-like variables, called compensators or also tags, multiplied by the first integral $u = x^q y^p$ of the linear part. Next we consider the unfolding of such a resonance, and provide (weaker) Gevrey-type linearization using compensators.

1 Introduction

Let X be a planar vector field with a saddle type $p : -q$ resonant singularity at $(0,0)$, where p and q are positive integers with $\gcd(p,q) = 1$. With this we mean that the linear part of X at $(0,0)$ has eigenvalues $\lambda > 0 > \mu$ with a ratio $\lambda/\mu = -p/q$. When X is C^∞ near $(0,0)$, there exists a C^∞ change of variables (a *conjugacy*) putting X in the following normal form:

$$Y(x,y) = x(\lambda + F(u))\frac{\partial}{\partial x} + y(\mu + G(u))\frac{\partial}{\partial y} \tag{1}$$

where $u = x^q y^p$ is the first integral of the linear part, where $(F,G)(u) = O(u)$; see for instance [8].

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Let us define the notion of Gevrey asymptotics for a formal power series:

Definition 1. *Let $n \geq 1$ be an integer. For $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ and $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$ we use the habitual multi-index notations $x^k = x_1^{k_1} \ldots x_n^{k_n}$, $k! = k_1! \ldots k_n!$ and $|k| = k_1 + \cdots + k_n$. Let $(a_k)_{k \in \mathbb{N}^n}$ be a sequence in some normed space.*

Let $s \geq 0$. We say that a formal power series $\sum_{k \in \mathbb{N}^n} a_k x^k$ is of Gevrey order s if there exist $C, r > 0$ such that $|a_k| \leq C k!^s r^{|k|}$ for all $k \in \mathbb{N}^n$.

Remark that for $s = 0$ the series has a strictly positive radius of convergence, that is: it defines an analytic function near 0.

Throughout this paper we will often need [6, Theorem 2.4]. For the convenience of the reader we state this here once more. With $\mathbb{N}^n_{quad}$ we mean all the n-tuples $k = (k_1, \ldots, k_n)$ with $|k| \geq 2$.

Theorem 1. *Given is a formal vector field*

$$\hat{X}_\delta : \dot{x} = A_\delta x + \sum_{|k| \geq 2} f_k(\delta) x^k$$

where:
(i) $\delta \in \Lambda$ where Λ is some set of parameters,
(ii) A_δ is the diagonal matrix $A_\delta = \mathrm{diag}[\mu_1(\delta), \ldots, \mu_n(\delta)]$,
(iii) the coefficient functions f_k are in some subalgebra of the algebra of bounded functions $\Lambda \to \mathbb{C}^n$, equipped with a complete multiplicative norm.

Assume that this series is of Gevrey order s for some $s \in [0, 1]$. Let $\mathcal{B} \subset \{1, \ldots, n\} \times \mathbb{N}^n_{quad}$ and let $\mathcal{G}$ be its complement. We make the following hypothesis:

$$\exists K > 0, \forall \delta \in \Lambda, \forall (j, k) \in \mathcal{G} : |\langle (\mu_1(\delta), \ldots, \mu_n(\delta)), k \rangle - \mu_j(\delta)| \geq K |k|^{1-s}. \quad (2)$$

Then there exists a formal power series transformation

$$\hat{\psi}_\delta(x) = x + \sum_{|k| \geq 2} u_k(\delta) x^k$$

of Gevrey order s conjugating X_δ to

$$\hat{Y}_\delta : \dot{x} = A_\delta x + \sum_{|k| \geq 2} g_k(\delta) x^k$$

with the property: $(j, k) \in \mathcal{G} \Rightarrow g_{k,j} = 0$, that is: only monomials with index inside $\mathcal{B}$ may appear in the normal form $\hat{Y}_\delta$.

Remark 1. Here we recall some classical facts that will be a commonly used throughout this article.

(i) Several results will be about formal power series. A theorem of Borel states that every formal power series can be realized as the Taylor series of a (non-unique) C^∞ function, and for Gevrey series additional properties of such a 'realizing' function can be obtained (Borel-Ritt theorem). For a proof, and for further basic information about Gevrey series, we refer the reader to [14, 2, 1].

(ii) Concerning conjugacies: when $\hat{Y} = \hat{\psi}^*\hat{X}$ (that is: the vector field $\hat{X}$ is conjugated to $\hat{Y}$ by the transformation $\hat{\psi}$), where $\hat{X}$, $\hat{Y}$ and $\hat{\psi}$ are formal power series, we can thus consider C^∞ realizations X, Y respectively ψ. Then $Y = \psi^*X + R_\infty$, where R_∞ has a vanishing Taylor series. In case that X is hyperbolic at 0 (which will always be the case in this paper), the 'flat remainder' R_∞ can be removed by a C^∞ transformation infinitely tangent to the identity. For a proof, see for example [8]. In case that $\hat{X}$, $\hat{Y}$ and $\hat{\varphi}$ have a positive radius of convergence (that is: they are analytic near 0) the flat remainder R_∞ is also analytic near 0 and is hence identically zero. So in that case we have $Y = \psi^*X$.

Let X be a planar vector field with a saddle type $p : -q$ resonant singularity as in the beginning of this introduction, and let Y be its normal form (1). As a corollary of Theorem 1 one has:

Theorem 2. *If X has a Taylor series, at $(0,0)$, that is of Gevrey order 1, then the transformation into the normal form Y is also of Gevrey order 1.*

In Theorem 6 and section 5 we will come back to this result, and will explain why this is indeed a corollary. Observe that if X is in particular analytic, then its Taylor series is surely of Gevrey order 1, so Theorem 2 can be applied; nevertheless, it is 'exceptional' that this transformation is convergent [12].

Example 1. The result in Theorem 2 is 'optimal': for the (even polynomial) vector field

$$X(x,y) = x(1 - \frac{1}{2}xy)\frac{\partial}{\partial x} + y(-1 - \frac{1}{2}xy + x^2y)\frac{\partial}{\partial y},$$

with a $1 : -1$ resonance, it appears that the normalizing transformation contains formal series of the form $\sum_{n\geq 1} n!z^n$. (Thanks to Peter De Maesschalck for this example.)

In this paper we want to remain as much as possible in this Gevrey category. Generically, one can further reduce X to a polynomial normal form by a C^∞ transformation [13, 17, 11], but we ignore if this can be done Gevrey; this is not the subject of this paper.

In the sequel we shall thus start from a vector field of the form (1) which we rename again X and where the series of $(F,G)(u)$ is of Gevrey order 1 in (x,y). As $u = x^q y^p$, this means that $(F,G)(u)$ is of Gevrey order $d := q + p$ in the variable u. In the next result, stated below in Theorem 3, the value of $d \geq 0$ will be arbitrary (and is not necessarily equal to $q + p$, although this would be a 'natural' choice).

The terms in F and G are usually called *resonant terms*; in general they cannot be 'transformed away' by a smooth (even C^2) change of variables. However, it is known that there exists a so called Logarithmic Mourtada Type (LMT) homeomorphism that linearizes X: see [7] for a definition of LMT in a general context and for a proof. However here we won't need that general definition and it will suffice to state that the linearizing transformation $(x,y) = \psi(\hat{x},\hat{y})$ will be of the form

$$(x,y) = \mathcal{H}(\hat{x},\hat{y},\log(|\hat{x}|)\hat{x},\log(|\hat{y}|)\hat{y}) \tag{3}$$

where $\mathcal{H}$ is of class C^∞. Let us denote $\hat{u} = \hat{x}^q \hat{y}^p$. In Theorem 3 we will obtain that $\mathcal{H}$ is the following form:

$$\begin{cases} x = \hat{x}(1 + \chi_1(\log(|\hat{x}|)\hat{u}, \log(|\hat{y}|)\hat{u}, \hat{u})) \\ y = \hat{y}(1 + \chi_2(\log(|\hat{x}|)\hat{u}, \log(|\hat{y}|)\hat{u}, \hat{u})) \end{cases} \tag{4}$$

and we will prove that (χ_1, χ_2) has a Gevrey-d series in its variables.

In a *family* of planar vector fields, the ratio of the eigenvalues can change when the parameters vary. Generically, we need one parameter, say ε close to zero, to unfold the resonance. We will study this unfolding, from different viewpoints.

One approach is to consider the parameter as an extra variable. In this case the linear part is $\lambda x\partial/\partial x + \mu y\partial/\partial y + 0.\partial/\partial\varepsilon$, and is thought as 'fixed'. We shall explain why one can apply directly Theorem 1 and obtain the usual formal normal form with a formal power series that is Gevrey-1 in the variables (x, y, ε).

Another viewpoint is to consider the normal form obtained by perturbing the linear part, which is then in essence of the form $\alpha(\varepsilon).((p+\varepsilon)x\partial/\partial x - qy\partial/\partial y)$ with $\alpha(0) \neq 0$; see section 2 for explanation. In this case, the condition needed to apply the result in Theorem 1 becomes fragile when the parameter ε varies, notably the hypothesis in (2). We provide an adapted version, taking into account this issue, and where series in (x, y) are used in which the coefficients are (for instance analytic) functions of the parameter. The domains of these functions will depend on the degree and will likely shrink inversely proportional to the degree: see section 2 (especially Theorem 5) for more precision. We comment on such type of series in Remark 4.

In Theorem 7 we extend the ideas of Theorem 3, about the removal of resonant terms, to the presence of the unfolding parameter ε. In this case, we shall have to unfold the logarithm function to Roussarie-Écalle compensators denoted s_ε, t_ε: see (27), (28). In accordance with Theorem 3 for the non-parametric case, we infer asymptotic properties of the linearizing transformation in sections 6.3 and 6.4. The estimates are summarized in Theorem 7, and we were only able to obtain a Gevrey order one worse than in the non-parametric case in Theorem 3.

It is unclear whether we can sharpen this result or not: in section 6.3 we use the majorant method leading to rather involved upper estimates, see Remark 9 in section 6.4. So far we were unable to construct examples showing the optimality of Theorem 7.

In the above, we lose information when starting from an analytic (family of) vector field(s) with a $p : -q$ resonant singularity. In Theorem 8 we give an explicit expression of a 'near' normal form that can be obtained by analytic conjugacy. In this expression, apart from the usual development in resonant terms in $u = x^q y^p$, one inevitably also encounters terms with a nonzero resonance condition, but as 'flat' as desired. See Theorem 8 for precision.

Finally, in Theorem 9 we try out the method in [5], where linearization is carried out starting from the analytic vector field obtained like in Theorem 8, but this time with an unfolding parameter ε, and we give an explicit procedure. Compared to Theorem 7, we shall need a more involved (multi-)sequence of polynomial functions $(\omega_{K_n} : n \geq 1, K_n \in \mathbb{Z}^n)$ of the Roussarie-Écalle compensators. These polynomials are obtained in an algorithmic way and are 'universal' in the sense that they do not depend on the given vector field, only on (p, q).

A natural question is again about the asymptotic nature of the linearizing transformation. Although the procedure to obtain the linearization is also here a step by step method and is explicitly obtained from a recursion, it is more intricate than in Theorem 7 or than in [5]. So far we were unable to address this question in a satisfactory way.

2 Main results

We state the main theorems of this paper, and defer their proofs to later sections.

We start from a vector field X in normal form (1) like in the introduction:

Theorem 3. *Let*

$$X(x,y) = x(\lambda + F(u))\frac{\partial}{\partial x} + y(\mu + G(u))\frac{\partial}{\partial y} \tag{5}$$

with $\lambda/\mu = -p/q$, $\gcd(p,q) = 1$ *and* $u = x^q y^p$. *Assume that* (F,G) *is of Gevrey order* $d \geq 0$. *Then there exists a change of variables* $(x,y) = \psi(\hat{x},\hat{y})$ *of the form (4) such that* $\psi^* X(\hat{x},\hat{y}) = \lambda\hat{x}\partial/\partial\hat{x} + \mu\hat{y}\partial/\partial\hat{y}$. *Moreover* (χ_1,χ_2) *is of Gevrey order* d.

Remark 2. *In some specific circumstances [9, 10, 18, 19] it can happen that the normal form in (5) is analytic. In this case we can take* $d = 0$, *and from Theorem 3 it then follows that, in the expression (4) of the linearizing transformation* ψ, *the function* (χ_1,χ_2) *is analytic in its variables a neighbourhood of* $(0,0,0)$.

Next we consider the unfolding of a $p : -q$ resonance. More specifically, let X_δ be a family of vector fields near $(0,0)$, depending smoothly (see below for more precision) on a parameter δ close to zero, and assume that, for $\delta = 0$, there is a $p : -q$ resonant singularity in $(0,0)$. By the implicit function theorem we may, and will, assume that $X_\delta(0,0) = 0$ for all δ close to zero. Moreover, the eigenvalues of the linear part of X_δ at $(0,0)$ are functions $\lambda(\delta)$, $\mu(\delta)$ of the parameter δ with

$$\frac{\lambda(0)}{\mu(0)} = \frac{p}{-q}. \tag{6}$$

One approach is to consider the parameter δ as an extra variable near $\delta = 0$, and to examine the vector field

$$X_\delta(x,y) = (\lambda(0)x + O(|(x,y,\delta)|^2))\frac{\partial}{\partial x} + (\mu(0)y + O(|(x,y,\delta)|^2))\frac{\partial}{\partial y} + 0.\frac{\partial}{\partial \delta}. \tag{7}$$

The linear part, in this point of view, is

$$\lambda(0)x\frac{\partial}{\partial x} + \mu(0)y\frac{\partial}{\partial y} + 0.\frac{\partial}{\partial \delta}.$$

In the next theorem we will, among other things, recall the formal Poincaré-Dulac normal form, and in its proof we will thereby estimate, from below, the 'small denominators' appearing in the normalizing transformation.

Theorem 4. *If X_δ in (7) has a Taylor expansion that is Gevrey-1 in (x, y, δ), then there is a formal power series transformation in (x, y, δ) of type Gevrey-1 conjugating X_δ to the formal normal form*

$$x(\lambda(0) + F(\delta, u))\frac{\partial}{\partial x} + y(\mu(0) + G(\delta, u))\frac{\partial}{\partial y} + 0.\frac{\partial}{\partial \delta} \tag{8}$$

where $u = x^q y^p$ and where $(F, G)(\delta, u) = O(|(\delta, u)|)$.

In the study of this unfolding, we shall need some elementary facts from number theory [15], based on Euclid's algorithm:

Proposition 1. *Let p and q be integers with $\gcd(p, q) = 1$. There exists a unique $(r_0, s_0) \in \mathbb{N}^2$ with $0 \le r_0 \le q$ and $0 \le s_0 \le p$ such that*

$$\langle (r_0, s_0), (p, -q) \rangle = pr_0 - qs_0 = 1.$$

Let us denote $(r_1, s_1) = (q, p) - (r_0, s_0)$, as well as

$$L = \begin{pmatrix} q & r_0 \\ p & s_0 \end{pmatrix}$$

and

$$M = \begin{pmatrix} q & r_1 \\ p & s_1 \end{pmatrix}.$$

Then:
(i) $\langle (r_1, s_1), (p, -q) \rangle = -1$,
(ii) L and M define one-to-one maps $\mathbb{Z}^2 \to \mathbb{Z}^2$.

Proof. Statement (i) is trivial, and (ii) follows from the fact that the determinants of L and M are equal to -1 respectively 1. ∎

Statement (ii) above means that for each $(k_1, k_2) \in \mathbb{Z}^2$ there exist unique $(m_1, m_2) \in \mathbb{Z}^2$ such that

$$(k_1, k_2) = m_1(q, p) + m_2(r_0, s_0), \tag{9}$$

Examples: for a $1 : -1$ resonance with $(q, p) = (1, 1)$ we take $(r_0, s_0) = (1, 0)$, and for $(q, p) = (2, 5)$ we take $(r_0, s_0) = (1, 2)$.

Another approach is to consider the normal form obtained by perturbing the linear part of the vector field:

$$A_\delta(x, y) := \lambda(\delta)x\frac{\partial}{\partial x} + \mu(\delta)y\frac{\partial}{\partial y}, \tag{10}$$

always with the resonance hypothesis (6). Let us denote

$$\alpha = \frac{\lambda(0)}{p} = -\frac{\mu(0)}{q}. \tag{11}$$

Assuming that the eigenvalues vary in a Lipschitz way in δ we have

$$|(\lambda(\delta), \mu(\delta)) - (\lambda(0), \mu(0))| \leq \mathrm{L}.|\delta|$$

for some $\mathrm{L} \geq 0$.

For $k = (k_1, k_2) \in \mathbb{N}^2$ we consider 'typical' monomials $x^{k_1}y^{k_2}x\partial/\partial x$ and $x^{k_1}y^{k_2}y\partial/\partial y$. Let $(m_1, m_2) = L^{-1}(k_1, k_2)$ be as in (9). Monomials with $m_2 = 0$ are of the form $u^{m_1}x\partial/x$ or $u^{m_1}y\partial/\partial y$ and are called *resonant*; the other ones are called *nonresonant*.

In view of condition (2) we want to estimate the quantity $|\langle(k, (\lambda(\delta), \mu(\delta)\rangle|$ from below. We can write

$$\langle k, (\lambda(\delta), \mu(\delta)\rangle = \alpha m_2 - \langle k, (\lambda(0), \mu(0)) - (\lambda(\delta), \mu(\delta))\rangle,$$

where α is defined by (11). Hence in case of nonresonance, that is: $|m_2| \geq 1$, we can estimate form below:

$$|\langle k, (\lambda(\delta), \mu(\delta)\rangle| \geq |\alpha||m_2| - \mathrm{L}|k||\delta| \geq |\alpha| - \mathrm{L}|k||\delta|. \tag{12}$$

Remark that this last number might become zero or negative for large $|k||\delta|$. It is clear that we will have to confine $|\delta|$, depending on k. One classical approach is in [13]: given a 'wanted' degree of differentiability $M \in \mathbb{N}$, there exists $K > 0$, depending on M and on $(\lambda(0), \mu(0))$, and there exists $\delta_0 > 0$ such that X_δ is C^M conjugated to polynomial normal form of degree at most K in u whose coefficients are smooth functions of δ, $|\delta| < \delta_0$. We will not consider this here any further.

We now present the topic from another angle, and will reconsider Theorem 1 and its proof in [6, Theorem 2.4]. We modify the hypothesis H_s on page 380 of this paper and adapt it to the situation encountered above, that is: if $|k|$ increases then we have to confine the parameter δ to a smaller domain. From the inequality in (12) we see that the diameter of this domain decreases to zero proportional to $1/|k|$. In order to be self contained, we now redefine the concepts used in this result.

Notation 1. Let l be a positive integer (think of: $l = |k|$ in the above). In view of the estimate in (12), we consider, for the parameter δ, the ball $\Lambda_l = B(0, \frac{|\alpha|}{2\mathrm{L}l})$ and let $\mathcal{A}_l$ be a subalgebra of the algebra of bounded functions of δ on Λ_l. We equip this space with a complete multiplicative norm $|.|_l$. We furthermore assume that the following property holds:

$$f \in \mathcal{A}_l \text{ and } g \in \mathcal{A}_m \Rightarrow |f.g|_{l+m} \leq |f|_l.|g|_m. \tag{13}$$

Observe that for all $\delta \in \Lambda_l$ and for all $|k| = l$ we have the estimate $|\langle k, (\lambda(\delta), \mu(\delta)\rangle| \geq |\alpha|/2$. In order to fix the ideas of the reader, we could for example take $\mathcal{A}_l$ to be the space of bounded analytic functions on Λ_l, equipped with the sup-norm.

We consider a family of vector fields

$$\hat{X}_\delta : \dot{x} = A_\delta x + \sum_{|k|\geq 2} f_k(\delta)x^k \tag{14}$$

in n dimensions such that the coefficient functions satisfy $f_k \in \mathcal{A}_{|k|}$. We assume that A_δ is the diagonal matrix $A_\delta = \mathrm{diag}[\mu_1(\delta), \ldots, \mu_n(\delta)]$.

Definition 2. (Compare with the hypothesis in (2).) *Let* $s \geq 0$ *and* $\mathcal{G} \subset \{1, \ldots, n\} \times \mathbb{N}^n_{quad}$. *We say that* $\mathcal{G}$ *satisfies the hypothesis of order s* if

$$\exists K > 0, \forall (j,k) \in \mathcal{G}, \forall \delta \in \Lambda_{|k|} : |\langle (\mu_1(\delta), \ldots, \mu_n(\delta)), k\rangle - \mu_j(\delta)| \geq K|k|^{1-s}. \quad (15)$$

Definition 3. *We say that the series in (14) is of Gevrey order s if there exist* $C, r > 0$ *such that* $|f_k|_{|k|} \leq C|k|!^s r^{|k|}$.

We can almost repeat the proof of [6, Theorem 2.4] and obtain:

Theorem 5. *Let* $\hat{X}_\delta$ *be a formal vector field as in (14). Assume that the series in (14) is of Gevrey order s, with* $0 \leq s \leq 1$. *If* $\mathcal{G} \subset \{1, \ldots, n\} \times \mathbb{N}^n_{quad}$ *satisfies the hypothesis of order s (15), then there exists a formal power series*

$$\hat{\psi}_\delta(x) = x + \sum_{|k|\geq 2} u_k(\delta) x^k,$$

with $u_k \in \mathcal{A}_{|k|}$, *of Gevrey order s, conjugating* $\hat{X}_\delta$ *to*

$$\hat{Y}_\delta : \dot{x} = A_\delta x + \sum_{|k|\geq 2} g_k(\delta) x^k$$

with the property that $(j,k) \in \mathcal{G} \Rightarrow g_{k,j}(\delta) = 0$, *that is: only monomials with index in the complement of* $\mathcal{G}$ *may appear in the normal form* $\hat{Y}_\delta$.

Remark 3. Note that the property in (13) is used in the proof of [6, Proposition 2]. In fact, we could have made the sequence of spaces $\mathcal{A}_l$ in Theorem 5 more abstract, but we will not need this here.

Remark 4. Let us comment on series of the form $\sum_k f_k(\delta) x^k$ like in (14), where $f_k \in \mathcal{A}_{|k|}$. Let us consider the case where, for the integers $l \geq 2$, the space $\mathcal{A}_l$ consists of the bounded analytic functions of $|\delta| \leq D/l$ for some constant D, and $|.|_l$ denotes the sup norm. Now assume that we have, for $s \geq 1$, the Gevrey s-type estimate

$$|f_k|_{|k|} \leq C|k|!^s r^{|k|} \quad (16)$$

for some $C, r > 0$. Let us write $l = |k|$. As we assume f_k to be analytic on $|\delta| \leq D/l$ we can write it as a convergent power series $f_k(\delta) = \sum_{j=0}^{\infty} f_{kj}\delta^j$ with a radius of convergence more than D/l. We apply Cauchy's inequalities in the usual way:

$$\begin{aligned} |f_{kj}| &= |f_k^{(j)}(0)/j!| \\ &= |\frac{1}{2\pi i} \oint_{|z|=\frac{D}{l}} \frac{f_k(z)}{z^{j+1}} dz| \\ &\leq (\frac{l}{D})^j |f_k|_{|k|} \\ &\leq (\frac{l}{D})^j . C|k|!^s r^{|k|} \\ &= (\frac{l}{D})^j . C l!^s r^l. \end{aligned} \quad (17)$$

We claim that the series $\sum_{kl} f_{kl} x^k \delta^l$ is 'somehow better' than being Gevrey-s in (x,δ); what we mean by this is indicated below. Stirling's approximation for factorials yields

$$\sqrt{2\pi} l^{l+\frac{1}{2}} e^{-l} \leq l! \leq e.l^{l+\frac{1}{2}} e^{-l}.$$

In particular:

$$(l+j)!^s \geq (\sqrt{2\pi})^s (l+j)^{(l+j+\frac{1}{2})s} e^{-ls-js}.$$

We can estimate, by (17), for $j \geq 1$:

$$\begin{aligned} |f_{kj}| &\leq C\frac{1}{D^j} l^j (e.l^{l+\frac{1}{2}} e^{-l})^s r^l \\ &\leq Ce^s (e^{-s})^l \frac{1}{D^j} l^{js+(l+\frac{1}{2})s} r^l \qquad (18) \\ &< Ce^s (e^{-s})^l \frac{1}{D^j} (l+j)^{js+(l+\frac{1}{2})s} r^l \qquad (19) \\ &\leq Ce^s (e^{-s})^l \frac{1}{D^j} ((l+j)!^s \sqrt{2\pi}^{-s} e^{ls+js}) r^l \\ &= Ce^s \sqrt{2\pi}^{-s} (|k|+j)!^s r^{|k|} (D^{-1} e^s)^j. \qquad (20) \end{aligned}$$

Inequality (20) shows that $\sum_{kj} f_{kj} x^k \delta^j$ is of Gevrey-s order in (x,δ), but moreover in (19) we have a strict inequality that 'becomes stronger' since the base $l+j$ of the involved power increases when j increases. It would be interesting to investigate the possible significance of this observation, but we have no further clue for the moment.

Remark 5. It appears to be important that $s \geq 1$ in Remark 4, notice the estimate in (18). Indeed, consider for instance the case $s = 0$ and the following counterexample in dimension $n = 1$. Take $f_k(\delta) = \frac{1}{1-k\delta}$ and $D = 1/2$, thus $f_{kj} = k^j$. Then f_k is analytic on $|\delta| \leq 1/(2k)$ and $|f_k|_k = \sup_{|\delta| \leq 1/(2k)} |f_k(\delta)| = |f_k(1/(2\delta)| = 2$, and hence we have inequality (16) for $s = 0$ with $C = 2$ and $r = 1$. On the other hand, the series $\sum_{kj} k^j x^k \delta^j$ is not analytic near $(x,\delta) = (0,0)$.

Combining the requirements $s \geq 1$ and $0 \leq s \leq 1$ from Theorem 5 only leaves us $s = 1$, which fortunately will be the only case that we shall need and that will occur 'naturally' below.

Let us come back to the family X_δ with linear part A_δ as in (10), that is: it unfolds a $p : -q$ resonance. Generically, we only need one parameter to describe this unfolding, and use the parameter ε such that

$$\frac{\lambda(\delta)}{\mu(\delta)} = \frac{p+\varepsilon}{-q}.$$

or equivalently $\varepsilon = \varepsilon(\delta) = -q\frac{\lambda(\delta)}{\mu(\delta)} - p$; notice that $\varepsilon(0) = 0$.

We make the assumption that $\varepsilon(\delta)$ is a submersion, and hence pass to the parameter ε instead of δ. From here on we may, and do, assume that

$$\frac{\lambda(\varepsilon)}{\mu(\varepsilon)} = \frac{p+\varepsilon}{-q}.$$

Let us summarize here that we shall consider a family X_ε with linear part $A_\varepsilon = dX_\varepsilon(0)$ of the following form:

$$A_\varepsilon(x,y) = \alpha(\varepsilon).((p+\varepsilon)x\frac{\partial}{\partial x} - qy\frac{\partial}{\partial y}) \tag{21}$$

with $\alpha(\varepsilon) = -\mu(\varepsilon)/q$ and $\alpha(0) = -\mu(0)/q \neq 0$. First of all, we confine ε so that $\alpha(\varepsilon)$ stays away from zero, for example we let $\varepsilon_0 > 0$ be so that for all $|\varepsilon| < \varepsilon_0$:

$$|\alpha(\varepsilon)| \geq \frac{1}{2}|\alpha(0)|. \tag{22}$$

Definition 4. Let, for $l \geq 1$ an integer, $\mathcal{A}_l$ be a subalgebra of the algebra of bounded functions of ε on the ball $\Lambda_l = B(0, \frac{1}{2l}) \cap B(0, \varepsilon_0)$.

In the next theorem we will assume that the nonlinear part is Gevrey of order $s = 1$, which of course includes the analytic case. We shall also assume that the manifolds $x = 0$ and $y = 0$ are invariant. Thanks to the hyperbolicity this is no restriction: see for example [3, 6].

Theorem 6. *Let X_ε be a one-parameter family of planar vector fields of the form*

$$X_\varepsilon(x,y) = \alpha(\varepsilon).((p+\varepsilon)x\frac{\partial}{\partial x} - qy\frac{\partial}{\partial y}) + F(\varepsilon,x,y)x\frac{\partial}{\partial x} + G(\varepsilon,x,y)y\frac{\partial}{\partial y}$$

where $(F,G)(\varepsilon,x,y) = O(|(x,y)|)$ is Gevrey of order $s = 1$. Then there is a change of variables ψ_ε of Gevrey order 1 conjugating X_ε to

$$Y_\varepsilon(x,y) = \alpha(\varepsilon).((p+\varepsilon)x\frac{\partial}{\partial x} - qy\frac{\partial}{\partial y}) + x\sum_{l\geq 1} a_l(\varepsilon)u^l\frac{\partial}{\partial x} + y\sum_{l\geq 1} b_l(\varepsilon)u^l\frac{\partial}{\partial y} \tag{23}$$

where the functions a_l, b_l belong to $\mathcal{A}_{(q+p)l}$, and moreover, there exist $C, r > 0$ such that

$$|a_l|_{(q+p)l} \leq Cl!^{q+p}r^l;$$

similarly for b_l.

In the sequel we take the expression in (23) as a starting point. Let us rename Y_ε again X_ε, and let us abbreviate

$$(F^1, G^1)(\varepsilon, u) = \sum_{l\geq 1}(a_l, b_l)(\varepsilon)u^l.$$

Then this series is of Gevrey order $q+p$ in the variable u, and we can thus write

$$X_\varepsilon(x,y) = \alpha(\varepsilon).((p+\varepsilon)x\frac{\partial}{\partial x} - qy\frac{\partial}{\partial y}) + xF^1(\varepsilon,u)\frac{\partial}{\partial x} + yG^1(\varepsilon,u)\frac{\partial}{\partial y}. \tag{24}$$

Next we give a procedure, inspired by the one in Theorem 3, in order to linearize X_ε. The logarithm-like functions now shall have to be unfolded and will depend on the parameter ε: we will use Roussarie-Écalle compensators [16], see below for specific formulas and details.

Also the asymptotics will be given in terms of a Gevrey majorant. Although the ideas will be comparable to the ones used for Theorem 3, the computations and estimates shall turn out to be more involved.

We will need below a new variable s_ε such that

$$\frac{d}{dt}s_\varepsilon = 1 - \alpha(\varepsilon)q\varepsilon s_\varepsilon. \tag{25}$$

In order to fix the ideas, consider the linearized vector field

$$A_\varepsilon(\tilde{x},\tilde{y}) = \alpha(\varepsilon).((p+\varepsilon)\tilde{x}\frac{\partial}{\partial\tilde{x}} - q\tilde{y}\frac{\partial}{\partial\tilde{y}}); \tag{26}$$

we look for $s_\varepsilon = s_\varepsilon(\tilde{x})$ satisfying (25). A direct calculation shows that

$$s_\varepsilon = \frac{1-|\tilde{x}|^{\frac{-q\varepsilon}{p+\varepsilon}}}{\alpha(\varepsilon)q\varepsilon} \tag{27}$$

is a solution. Similarly, for $t_\varepsilon = s_\varepsilon(\tilde{y})$ we can take

$$t_\varepsilon = \frac{1-|\tilde{y}|^{\varepsilon}}{\alpha(\varepsilon)q\varepsilon}. \tag{28}$$

Remark 6. (i) Functions such as in (27) and (28) are sometimes referred to as *Roussarie-Écalle compensators*. They are encountered for instance in the calculation of the Dulac transition map near a saddle type hyperbolic singular point of a family of planar vector fields [16].
(ii) Often one is only interested in equivalence instead of conjugacy, meaning that changes of variables preserve orbits but not necessarily time for the vector fields. In that case, these compensator functions s_ε and t_ε can be simplified by taking $\alpha(\varepsilon) \equiv 1$.
(iii) An easy computation shows that the limit of the compensators when $\varepsilon \to 0$ for fixed $(\tilde{x},\tilde{y})$ are: $\lim_{\varepsilon\to 0} s_\varepsilon = \frac{1}{\alpha(0)p}\log|\tilde{x}|$ and $\lim_{\varepsilon\to 0} t_\varepsilon = \frac{-1}{\alpha(0)q}\log|\tilde{y}|$.

In the next theorem we use the symbol '$\preceq$' in the context of formal power series for: 'is majorated by'.

Theorem 7. *Let X_ε be of the form (24). There exists a formal change of variables $(x,y) = \hat{\psi}(\tilde{x},\tilde{y})$ determined by*

$$(\tilde{x},\tilde{y}) = (x,y) + (x\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}s_\varepsilon^k F^k(\varepsilon,u), y\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}t_\varepsilon^k G^k(\varepsilon,u)) \tag{29}$$

conjugating X_ε to A_ε in (26), where $(F^k,G^k)(\varepsilon,u) = O(u^k)$.

Moreover, assume that the series (F^1,G^1) in (24) is of Gevrey order $d \geq 0$ in u. Then the series in (29) has a majorant

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}s_\varepsilon^k F^k(\varepsilon,u) \quad &\preceq \quad s_\varepsilon\sum_{n\geq 1} n!^d u^n + \frac{s_\varepsilon^2}{2!}\sum_{n\geq 2} n!^d\left(\frac{1}{2^{d+1}}(n+1)n + \eta(n-1)\right)u^n \\ &\quad + \sum_{i\geq 0} u^i \sum_{k\geq 3}(s_\varepsilon u)^k \sum_{j=0}^{k-1} f_{ikj}\eta^j\end{aligned} \tag{30}$$

where

$$
\begin{aligned}
f_{ik0} &\leq \frac{3^{i+2k-1}}{2^{(k-1)(d+1)}}\frac{1}{i+1}(i+k)!^d \\
f_{ikj} &\leq (i+k)!^d 2^{-kd+d+jd+j}3^{i+2k-1}\frac{(k-1)!}{i+1}\ \textit{for}\ 1 \leq j \leq k-2 \\
f_{ik,k-1} &\leq (i+k)!^d 2^{i+k-1}\frac{1}{k}
\end{aligned}
\qquad (31)
$$

and where $\eta = |\alpha(\varepsilon)q\varepsilon|$. *In particular, the series in (30) is of Gevrey order* $d+1$ *in the variables* $(u, s_\varepsilon u)$. *Similar estimates hold for the series defining* $\tilde{y}$.

Remark 7. Apparently we have to take this Gevrey order $d+1$ because of the formula in (31), which contains a factor $\sim k!^{d+1}$; in the other formulas we have better estimates $\sim k!^d$.

Assume that the initial family X_ε consists of analytic vector fields (locally near $(0,0)$, see below for precision). When invoking Theorem 6 one has to weaken this information and we only had inferred Gevrey-type results. In this paragraph we explicate a kind of 'nearly' normal form that can be obtained by *analytic* conjugacy, thereby letting the first integral $u = x^q y^p$ of $dX_0(0,0)$ play a dominant role .

We reconsider the family X_ε with linear part (21), that is: the parameter ε unfolds the $p : -q$ resonance, and we assume analyticity in the following sense. Let X_ε have a Taylor series near $(0,0)$ of the form

$$
X_\varepsilon(x,y) = \alpha(\varepsilon).((p+\varepsilon)x\frac{\partial}{\partial x} - qy\frac{\partial}{\partial y})+ \sum_{k_1+k_2\geq 1} a_{1k_1k_2}(\varepsilon)x^{k_1}y^{k_2}.x\frac{\partial}{\partial x} + \sum_{k_1+k_2\geq 1} a_{2k_1k_2}(\varepsilon)x^{k_1}y^{k_2}.y\frac{\partial}{\partial y}, \qquad (32)
$$

where α is continuous and $\alpha(0) \neq 0$.

Notation 2. We will furthermore assume that the coefficient functions in ε, appearing in (32), are all in some algebra $\mathcal{A}(\varepsilon_0)$ of bounded continuous functions on $]-\varepsilon_0, \varepsilon_0[$, for some fixed $\varepsilon_0 > 0$, equipped with a complete multiplicative norm.

For example: $\mathcal{A}(\varepsilon_0)$ could be the space of bounded analytic functions equipped with the sup-norm.

In the next theorem 8 we will consider series like above that are locally analytic in (x,y). With this we mean by definition: there is a majorant series in (x,y), independent of ε, with a positive radius of convergence. (Note: the assumption in expression (32) that the manifolds $x = 0$ and $y = 0$ are invariant is no limitation, since it is well known that the stable and unstable manifolds are analytic; for instance [6] contains one more proof of this fact.)

The idea now is to rewrite the series in (32), using conjugacy, in a form that emphasizes the first integral $u = x^q y^p$ of $dX_0(0,0)$. The integer $K \geq 0$ in the next theorem will reflect, loosely speaking, the 'order of deviation' from the normal form; we will make this more precise below in Remark 8.

Theorem 8. *Let X_ε be a family of the form (32) where:*
(i) the coefficient functions in this series are in $\mathcal{A}(\varepsilon_0)$ (see Notation 2),
(ii) the series is locally analytic in (x,y).
Let (r_0,s_0) and (r_1,s_1) be defined as in Proposition 1. Denote $u = x^q y^p$.
Let $K \geq 0$ be a given integer. If ε_0 is sufficiently small, then there exists a locally analytic (in (x,y)) change of variables of the form

$$\psi_\varepsilon(x,y) = (x,y) + \sum_{k_1+k_2\geq 1}(xu_{1k_1k_2}(\varepsilon)x^{k_1}y^{k_2}, yu_{2k_1k_2}(\varepsilon)x^{k_1}y^{k_2}), \tag{33}$$

with coefficient functions in $\mathcal{A}(\varepsilon_0)$, conjugating X_ε into $Y_\varepsilon = \psi_\varepsilon^* X_\varepsilon$ taking the form

$$\begin{aligned} Y_\varepsilon(x,y) \;=\;& \alpha(\varepsilon).((p+\varepsilon)x\frac{\partial}{\partial x} - qy\frac{\partial}{\partial y}) \\ &+(\sum_{k\geq 0} F_k(\varepsilon,u)(x^{r_0}y^{s_0})^k u^{Kk} + \sum_{k\geq 1}\tilde{F}_k(\varepsilon,u)(x^{r_1}y^{s_1})^k u^{Kk})x\frac{\partial}{\partial x} \\ &+(\sum_{k\geq 0} G_k(\varepsilon,u)(x^{r_0}y^{s_0})^k u^{Kk} + \sum_{k\geq 1}\tilde{G}_k(\varepsilon,u)(x^{r_1}y^{s_1})^k u^{Kk})y\frac{\partial}{\partial y} \end{aligned} \tag{34}$$

where $F_k, \tilde{F}_k, G_k$ and $\tilde{G}_k$ are locally analytic in u and $F_k(\varepsilon,0) = \tilde{F}_k(\varepsilon,0) = G_k(\varepsilon,0) = \tilde{G}_k(\varepsilon,0) = 0$.

Remark 8. Recall the formal normal form like in (8); this is equal to the terms in (34) with $k=0$, that is:

$$\alpha(0)(px\frac{\partial}{\partial x} - qy\frac{\partial}{\partial y}) + F_0(\varepsilon,u)x\frac{\partial}{\partial x} + G_0(\varepsilon,u)y\frac{\partial}{\partial y},$$

and the remainder of the nonlinear expression in (34)

$$\begin{aligned} R_\varepsilon(x,y) \;:=\;& (\sum_{k\geq 1} F_k(\varepsilon,u)(x^{r_0}y^{s_0})^k u^{Kk} + \sum_{k\geq 1}\tilde{F}_k(\varepsilon,u)(x^{r_1}y^{s_1})^k u^{Kk})x\frac{\partial}{\partial x} \\ &+(\sum_{k\geq 1} G_k(\varepsilon,u)(x^{r_0}y^{s_0})^k u^{Kk} + \sum_{k\geq 1}\tilde{G}_k(\varepsilon,u)(x^{r_1}y^{s_1})^k u^{Kk})y\frac{\partial}{\partial y} \end{aligned}$$

is a finitely flat perturbation of this normal form, more specifically:

$$R_\varepsilon(x,y) = O(u^K) = O(x^{qK}y^{pK}).$$

Notice that this flatness is large, when K is large, simultaneously in *both* x and y. So a consequence of the theorem above is that this can be achieved analytically, see also [3].

Finally in this paper, we want to use the first integral $u = x^q y^p$ in the linearization process of a family X_ε as in Theorem 8, since we are close to $p:-q$ resonance. For that purpose, we start from expression (34) for $K=0$, and we may hence start from an analytic vector field (rename Y_ε again X_ε) of the form

$$\begin{aligned} X_\varepsilon(x,y) &= \alpha(\varepsilon).((p+\varepsilon)x\frac{\partial}{\partial x} - qy\frac{\partial}{\partial y}) \\ &\quad +(\sum_{k\geq 0} F_k(\varepsilon,u)(x^{r_0}y^{s_0})^k + \sum_{k\geq 1} \tilde{F}_k(\varepsilon,u)(x^{r_1}y^{s_1})^k)x\frac{\partial}{\partial x} \\ &\quad +(\sum_{k\geq 0} G_k(\varepsilon,u)(x^{r_0}y^{s_0})^k + \sum_{k\geq 1} \tilde{G}_k(\varepsilon,u)(x^{r_1}y^{s_1})^k)y\frac{\partial}{\partial y}. \end{aligned} \tag{35}$$

We want to rewrite the occurring summations as one sum $\sum_{k\in\mathbb{Z}}$ as follows. Let us use notations

$$\begin{aligned} \tilde{F}_k &=: F_{-k} \\ \tilde{G}_k &=: G_{-k} \\ v_k &:= \begin{cases} (x^{r_0}y^{s_0})^k \text{ if } k\geq 0 \\ (x^{r_1}y^{s_1})^{-k} \text{ if } k<0, \end{cases} \end{aligned} \tag{36}$$

where (r_0,s_0) and (r_1,s_1) are as in Proposition 1. This way we can rewrite (35) in the following manner:

$$X_\varepsilon(x,y) = \alpha(\varepsilon).((p+\varepsilon)x\frac{\partial}{\partial x} - qy\frac{\partial}{\partial y}) + \sum_{k\in\mathbb{Z}} v_k F_k(\varepsilon,u)x\frac{\partial}{\partial x} + \sum_{k\in\mathbb{Z}} v_k G_k(\varepsilon,u)y\frac{\partial}{\partial y}. \tag{37}$$

We consider again the variables s_ε and t_ε satisfying equation (25); they unfold the logarithm function, see Remark 6. We have to consider these variables as 'large' for x close to 0. On the other hand, when multiplied by x or y, they become small, essentially like $x\log|x|$ or $y\log|y|$.

Notation 3.

- We denote

$$K_n = (k_1,\ldots,k_{n-1},k_n) = (K_{n-1},k_n) \in \mathbb{Z}^n$$

  and define inductively, starting from (36):

$$v_{K_n} = v_{K_{n-1}}.v_{k_n}. \tag{38}$$

- With $\sum_{K_n}$ we mean: a summation over all $K_n \in \mathbb{Z}^n$.
- For $k\in\mathbb{Z}$ we use the symbol

$$\delta_k = \begin{cases} = 0 \text{ for } k\geq 0 \\ = 1 \text{ for } k<0. \end{cases} \tag{39}$$

- For $K_n \in \mathbb{Z}^n$ we denote

$$\delta_{K_n} = (\delta_{k_1},\ldots,\delta_{k_n}),$$

  $D(K_n) = -\langle \delta_{K_n}, K_n\rangle + n$ and $|K_n| = |k_1| + \cdots + |k_n|$.

Theorem 9. *Let X_ε be of the form (37). There exists a formal change of variables $(x,y) = \hat{\psi}(\tilde{x},\tilde{y})$ determined by*

$$(\tilde{x},\tilde{y}) = (x,y) + \sum_{n=1}^{\infty}(-1)^n \sum_{K_n} \omega_{K_n}(s) v_{K_n}(xF_{K_n}(\varepsilon,u), yG_{K_n}(\varepsilon,u)) \tag{40}$$

where $s = s_\varepsilon$ or $s = t_\varepsilon$, linearizing X_ε, and with the following properties:
(i) $(F_{K_n}, G_{K_n})(\varepsilon,u) = O(u^n)$,
(ii) ω_{K_n} is a polynomial of degree at most $D(K_n)$,
(iii) for a term in the summation (40) we can write

$$\omega_{K_n}(s) v_{K_n}(F_{K_n}, G_{K_n})(u,\varepsilon) = P_{K_n,\varepsilon}(sx,sy).E_{K_n,\varepsilon}(x,y) \tag{41}$$

where
(a) $P_{K_n,\varepsilon}$ is a polynomial of degree at most $D(K_n)$, with coefficients that are analytic functions of ε, and
(b) $E_{K_n,\varepsilon}$ is a power series with coefficients that are analytic functions of ε.

Moreover, the polynomials ω_{K_n} only depend on p and q, and the multi-sequences (F_{K_n}, G_{K_n}) and ω_{K_n} can be generated recursively (see the explicit formulas in sections 8.2 and 8.3).

3 Proof of Theorem 3

In order to make the exposition self-contained we recall, following [7], briefly how to obtain $\mathcal{H}$, see the transformation in (3).

First of all we calculate that

$$u' = u(qF(u) + pG(u)). \tag{42}$$

Let us denote $H(u) = qF(u) + pG(u)$, then $u' = uH(u)$.

We introduce two additional variables s and t with the property that $s' = t' = 1$. These variables are sometimes referred to as *tags*. The meaning of these tags will become clear below.

3.1 Step one

We consider the new variables

$$(x_1,y_1) = (x - xsF(u), y - ytG(u)); \tag{43}$$

then, for the first component we can calculate that

$$x_1' = x_1\lambda - xs(F(u).F(u) + F'(u)uH(u)).$$

Similar computations for the second component of (43). If we denote

$$(F^2,G^2)(u) = (F(u).F(u) + F'(u)uH(u), G(u).G(u) + G'(u)uH(u)) = O(u^2)$$

then

$$(x_1', y_1') = (\lambda x_1, \mu y_1) - (xsF^2(u), ytG^2(u)).$$

3.2 Induction step and formal limit

Assume, by induction, that we have for $n \geq 2$:

$$(x'_{n-1}, y'_{n-1}) = (\lambda x_{n-1}, \mu y_{n-1}) + \frac{(-1)^{n-1}}{(n-1)!}(xs^{n-1}F^n(u), yt^{n-1}G^n(u)) \tag{44}$$

where $(F^n, G^n)(u) = O(u^n)$. We consider the new variables

$$(x_n, y_n) = (x_{n-1}, y_{n-1}) + \frac{(-1)^n}{n!}(xs^n F^n(u), yt^n G^n(u)). \tag{45}$$

Let us concentrate on the first component of (45), since the second one is completely similar. We get, after a short computation:

$$x'_n \quad = \quad \lambda x_n + \frac{(-1)^n}{n!} xs^n (F(u)F^n(u) + (F^n)'(u)uH(u)).$$

Therefore, if we denote

$$(F^{n+1}, G^{n+1})(u) = (F(u)F^n(u) + (F^n)'(u)uH(u), G(u)G^n(u) + (G^n)'(u)uH(u)), \tag{46}$$

the change of variables (45) transforms the equations (44) into

$$(x'_n, y'_n) = (\lambda x_n, \mu y_n) + \frac{(-1)^n}{n!}(xs^n F^{n+1}(u), yt^n G^{n+1}(u)).$$

Also: $(F^{n+1}, G^{n+1})(u) = O(u^{n+1})$. Observe that

$$(x_n, y_n) = (x, y) + \sum_{k=1}^{n} \frac{(-1)^k}{k!}(xs^k F^k(u), yt^k G^k(u)). \tag{47}$$

We take the formal limit $n \to \infty$ of (47):

$$(x_\infty, y_\infty) := (x, y) + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}(xs^k F^k(u), yt^k G^k(u)) =: \Psi(x, y, s, t). \tag{48}$$

Let us substitute $(s, t) = (\lambda^{-1} \log |x_\infty|, \mu^{-1} \log |y_\infty|)$ in (48) and consider the equation

$$(x_\infty, y_\infty) = \Psi(x, y, \lambda^{-1} \log |x_\infty|, \mu^{-1} \log |y_\infty|).$$

Let us also denote $u_\infty = x_\infty^q y_\infty^p$. We introduce extra variables

$$\begin{aligned} (\xi, \xi_1, \eta, \eta_1) &:= (sx, sx_\infty, ty, ty_\infty), \\ (\sigma, \sigma_1, \tau, \tau_1) &:= (su, su_\infty, tu, tu_\infty); \end{aligned}$$

then we can write Ψ in the form

$$\Psi(x, y, s, t) = (x + x\psi_1(\xi x^{q-1} y^p, u), y + y\psi_2(\eta x^q y^{p-1}, u)) = (x + x\psi_1(\sigma, u), y + y\psi_2(\tau, u)) \tag{49}$$

where ψ_1, ψ_2 are $O(1)$. With these notations we have

$$\begin{aligned}(\xi_1, \eta_1) &= (sx_\infty, ty_\infty) = (sx + sx\psi_1(\xi x^{q-1}y^p, u), ty + ty\psi_2(\eta x^q y^{p-1}, u)) \\ &= (\xi + \xi\psi_1(\xi x^{q-1}y^p, u), \eta + \eta\psi_2(\eta x^q y^{p-1}, u)).\end{aligned}$$

Then

$$\begin{cases} u_\infty = u(1+\psi_1(\sigma,u))^q(1+\psi_2(\tau,u))^p \\ \sigma_1 = \sigma(1+\psi_1(\sigma,u))^q(1+\psi_2(\tau,u))^p \\ \tau_1 = \tau(1+\psi_1(\sigma,u))^q(1+\psi_2(\tau,u))^p \end{cases} \tag{50}$$

and

$$\begin{cases} x_\infty = x + x\psi_1(\xi x^{q-1}y^p, u) \\ y_\infty = y + y\psi_2(\eta x^q y^{p-1}, u) \\ \xi_1 = \xi + \xi\psi_1(\xi x^{q-1}y^p, u) \\ \eta_1 = \eta + \eta\psi_2(\eta x^q y^{p-1}, u). \end{cases} \tag{51}$$

Using (50) and the inverse function theorem, for formal power series, we can write

$$\begin{cases} u = u_\infty(1+\varphi(\sigma_1,\tau_1,u_\infty)) \\ \sigma = \sigma_1(1+\varphi(\sigma_1,\tau_1,u_\infty)) \\ \tau = \tau_1(1+\varphi(\sigma_1,\tau_1,u_\infty)) \end{cases} \tag{52}$$

for some formal power series φ of order $O(1)$. We write (51) as follows:

$$\begin{cases} x = x_\infty(1+\psi_1(\sigma,u))^{-1} \\ y = y_\infty(1+\psi_2(\tau,u))^{-1} \\ \xi = \xi_1(1+\psi_1(\sigma,u))^{-1} \\ \eta = \eta_1(1+\psi_2(\tau,u))^{-1}, \end{cases}$$

and in the right hand side of the above we can subsequently substitute (u, σ, τ) by their power series from (52): we can write

$$(1+\psi_1(\sigma,u))^{-1} = 1+\chi_1(\sigma_1,\tau_1,u_\infty) \text{ and } (1+\psi_2(\tau,u))^{-1} = 1+\chi_2(\sigma_1,\tau_1,u_\infty) \tag{53}$$

for some power series χ_1, χ_2 of order $O(1)$. We obtain

$$\begin{cases} x = x_\infty(1+\chi_1(\sigma_1,\tau_1,u_\infty)) = x_\infty(1+\chi_1(su_\infty,tu_\infty,u_\infty)) \\ \qquad\qquad\qquad\qquad\qquad\quad = x_\infty(1+\chi_1(\xi_1 x_\infty^{q-1}y_\infty^p, \eta_1 x_\infty^q y_\infty^{p-1}, u_\infty)) \\ y = y_\infty(1+\chi_2(\sigma_1,\tau_1,u_\infty)) = y_\infty(1+\chi_2(su_\infty,tu_\infty,u_\infty)) \\ \qquad\qquad\qquad\qquad\qquad\quad = y_\infty(1+\chi_2(\xi_1 x_\infty^{q-1}y_\infty^p, \eta_1 x_\infty^q y_\infty^{p-1}, u_\infty)) \\ \xi = \xi_1(1+\chi_1(\sigma_1,\tau_1,u_\infty)) = \xi_1(1+\chi_1(su_\infty,tu_\infty,u_\infty)) \\ \qquad\qquad\qquad\qquad\qquad\quad = \xi_1(1+\chi_1(\xi_1 x_\infty^{q-1}y_\infty^p, \eta_1 x_\infty^q y_\infty^{p-1}, u_\infty)) \\ \eta = \eta_1(1+\chi_2(\sigma_1,\tau_1,u_\infty)) = \eta_1(1+\chi_2(su_\infty,tu_\infty,u_\infty)) \\ \qquad\qquad\qquad\qquad\qquad\quad = \eta_1(1+\chi_2(\xi_1 x_\infty^{q-1}y_\infty^p, \eta_1 x_\infty^q y_\infty^{p-1}, u_\infty)). \end{cases}$$

Let us abbreviate the first two lines by writing $(x,y) = \hat{\mathcal{H}}(x_\infty, y_\infty, \xi_1, \eta_1)$. Let $\mathcal{H}$ be a C^∞ map such that its Taylor series is equal to $\hat{\mathcal{H}}$ (see Remark 1 (i)). Then the transformation

$$(x,y) = \mathcal{H}(x_\infty, y_\infty, \lambda^{-1}\log|x_\infty|.x_\infty, \mu^{-1}\log|y_\infty|.y_\infty)$$

conjugates X to the linear vector field $A := \lambda x_\infty \partial/\partial x_\infty + \mu y_\infty \partial/\partial y_\infty$ up to an 'infinitely flat remainder' term R, that is: with a Taylor series $j_\infty R(0,0,0,0) = 0$. We lift A to these four variables, and remark that $s' = \lambda^{-1}\frac{1}{x_\infty}x'_\infty = 1$ and hence $\xi'_1 = sx'_\infty + x_\infty = \lambda\xi_1 + x_\infty$; similarly $\eta'_1 = \mu\eta_1 + y_\infty$. This linear vector field is hyperbolic, and R can be removed by a C^∞ transformation with at Taylor series equal to the identity (see Remark 1 (ii)).

3.3 Gevrey order of (χ_1, χ_2)

Now we prove that (χ_1, χ_2) is of Gevrey order d. Since we assume that F is of Gevrey order d, we have a majorant of the following form:

$$F(u) \preceq \sum_{k\geq 1} Ck!^d r^k u^k.$$

Although this is not essential, we shall pass to new rescaled variables $(\tilde{x}, \tilde{y}) = (\rho x, \rho y)$ in order to make the estimates and formulas in the sequel more concise, for some $\rho > 0$ to be determined presently. We denote

$$\tilde{u} = \tilde{x}^q \tilde{y}^p = \rho^{q+p} x^q y^p = \rho^{q+p} u.$$

Then

$$\tilde{x}' = \tilde{x}\left(\lambda + F\left(\frac{1}{\rho^{q+p}}\tilde{u}\right)\right).$$

We consider the majorant

$$F\left(\frac{1}{\rho^{q+p}}\tilde{u}\right) \preceq \sum_{k\geq 1} C\left(\frac{r}{\rho^{q+p}}\right)^k k!^d \tilde{u}^k. \tag{54}$$

By taking $\rho > 0$ large enough, and since $q + p > 0$, we can take care that

$$\frac{r}{\rho^{q+p}} \leq \min\left\{1, \frac{1}{C}\right\}$$

and we can continue (54) by:

$$F\left(\frac{1}{\rho^{q+p}}\tilde{u}\right) \preceq \sum_{k\geq 1} C\frac{r}{\rho^{q+p}} k!^d \tilde{u}^k \preceq \sum_{k\geq 1} k!^d \tilde{u}^k. \tag{55}$$

In a completely similar way we can treat G and obtain a majorant of the form (55). Concerning the function $H(u) = qF(u) + pG(u)$, like in (42), we compute easily that

$$\tilde{u}' = \tilde{u}\left(qF\left(\frac{\tilde{u}}{\rho^{q+p}}\right) + pG\left(\frac{\tilde{u}}{\rho^{q+p}}\right)\right).$$

In an analogous manner as above we can obtain a majorant

$$qF\left(\frac{\tilde{u}}{\rho^{q+p}}\right) + pG\left(\frac{\tilde{u}}{\rho^{q+p}}\right) \preceq \sum_{k\geq 1} k!^d \tilde{u}^k.$$

We conclude here that, up to a rescaling, we may, and shall, omit the tilde˜symbol and assume that $F(u)$, $G(u)$ and $H(u)$ have the majorant $\sum_{k\geq 1} k!^d u^k$ for some $d \geq 0$.

We now look for a majorant of the $F^k(u)$ in (48), using the relation (46). Let us write $F^1(u) = F(u)$ and assume, by induction on N, that we have a majorant of the form

$$F^N(u) \preceq \sum_{n\geq N} n!^d a_{nN} u^n. \tag{56}$$

We have the initial conditions $a_{n1} = 1$ for all $n \geq 1$. Recall the recursion

$$F^{N+1}(u) = F^N(u).F(u) + (F^N)'(u).u.H(u). \tag{57}$$

We remind of some useful inequalities concerning factorials:

Lemma 1. *Let $k = (k_1, \dots, k_p) \in \mathbb{N}^p$ and denote $|k| = k_1 + \cdots + k_p$.*

- *We have:*

$$|k|! \leq p^{|k|} k_1! \dots k_p!. \tag{58}$$

- *If all $k_i \geq 1$ then*

$$k_1! \dots k_p! \leq \frac{1}{p!}|k|!. \tag{59}$$

Let us continue with the proof of Theorem 3. We consider the first term in the right hand side of (57) and majorate, also using formula (59) with $p = 2$:

$$\begin{aligned} F^N(u).F(u) &\preceq \sum_{n\geq N} n!^d a_{nN} u^n . \sum_{k\geq 1} k!^d u^k \\ &\preceq \frac{1}{2^d} \sum_{m\geq N+1} m!^d u^m \sum_{\substack{n\geq N, k\geq 1 \\ n+k=m}} a_{nN} \\ &= \frac{1}{2^d} \sum_{m\geq N+1} m!^d u^m \sum_{n=N}^{m-1} a_{nN} \end{aligned} \tag{60}$$

and for the second term in (57):

$$\begin{aligned} (F^N)'(u).u.H(u) &\preceq \sum_{n\geq N} n.n!^d a_{nN} u^{n-1}.u. \sum_{k\geq 1} k!^d u^k \\ &\preceq \frac{1}{2^d} \sum_{m\geq N+1} m!^d u^m \sum_{n=N}^{m-1} n a_{nN}. \end{aligned}$$

Therefore

$$F^{N+1}(u) \preceq \frac{1}{2^d} \sum_{m\geq N+1} m!^d u^m \sum_{n=N}^{m-1} (1+n) a_{nN}.$$

Let us hence define inductively, for $m \geq N+1$:

$$a_{m,N+1} = \frac{1}{2^d} \sum_{n=N}^{m-1} (n+1) a_{nN} \tag{61}$$

with $a_{n,1} = 1$, and let us estimate the solution of the recursion equation (61).

Lemma 2. *For all $P \geq K$ we have*

$$\sum_{n=K}^{P}(n-1)(n-2)\dots(n-K+1) = \frac{1}{K}P.(P-1)\dots(P-K+1). \qquad (62)$$

Proof. Use induction on P. ∎

Lemma 3. *The solution of (61) is estimated by*

$$a_{m,N+1} \leq \frac{1}{2^{N(d+1)}}\frac{1}{N!}(m+N)(m+N-1)\dots(m+1)m\dots(m-N+1) = \frac{1}{2^{N(d+1)}}\frac{(m+N)!}{N!(m-N)!} \qquad (63)$$

for all $m \geq N+1$.

Proof. For $N=1$ we observe that

$$\begin{aligned} a_{m2} &= \frac{1}{2^d}\sum_{n=1}^{m-1}(n+1) \\ &\leq \frac{1}{2^d}\sum_{n=2}^{m+1}(n-1) \\ &= \frac{1}{2^d}\frac{1}{2}(m+1)m. \end{aligned}$$

Assume by induction that the estimate is true for N, then for all $m \geq N+2$:

$$\begin{aligned} a_{m,N+2} &= \frac{1}{2^d}\sum_{n=N+1}^{m-1}(n+1)a_{n,N+1} \\ &\leq \frac{1}{2^d}\sum_{n=N+1}^{m-1}(n+1)\frac{1}{2^{N(d+1)}}\frac{1}{N!}(n+N)(n+N-1)\dots \\ &\qquad (n+1)n\dots(n-N+1) \\ &\leq \frac{1}{2^{N(d+1)+d}}\frac{1}{N!}\sum_{n=N+1}^{m-1}(n+N+1)(n+N)\dots \\ &\qquad (n+1)n\dots(n-N+1) \\ &\leq \frac{1}{2^{N(d+1)+d}}\frac{1}{N!}\sum_{n=2N+2}^{m+N+1}(n-1)(n-2)\dots(n+1)n\dots(n-2N-1) \\ &= \frac{1}{2^{(N+1)(d+1)}}\frac{1}{(N+1)!}(m+N+1)(m+N)\dots(m-N) \qquad (64) \end{aligned}$$

in which we have applied formula (62) with $K = 2N+2$ and $P = m+N+1$. Thus we have formula (63) where N is replaced by $N+1$. ∎

From (56) and Lemma 3 we conclude that

$$F^N(u) \preceq \frac{1}{2^{(N-1)(d+1)}(N-1)!}\sum_{n\geq N} n!^d\frac{(n+N-1)!}{(n-N+1)!}u^n.$$

This is also a majorant for $G^N(u)$. Let us hence consider the majorant, for the series in (48), given by

$$S(u,s) := \sum_{k\geq 1} \frac{s^k}{k!} \frac{1}{2^{(k-1)(d+1)}(k-1)!} \sum_{n\geq k} n!^d \frac{(n+k-1)!}{(n-k+1)!} u^n,$$

and let us denote

$$\gamma_{nk} = \frac{n!^d (n+k-1)!}{k! 2^{(k-1)(d+1)}(k-1)!(n-k+1)!};$$

then

$$\begin{aligned} S(u,s) &= \sum_{k\geq 1}\sum_{n\geq k} \gamma_{nk} u^n s^k \\ &= \sum_{k\geq 1}\sum_{i\geq 0} \gamma_{i+k,k} u^i (su)^k. \end{aligned}$$

We estimate $\gamma_{i+k,k}$, also using formula (58) with $p = 3$:

$$\begin{aligned} \gamma_{i+k,k} &= \frac{(i+k)!^d (i+2k-1)!}{k! 2^{(k-1)(d+1)}(k-1)!(i+1)!} \\ &\leq \frac{(i+k)!^d 3^{i+2k-1} i! k! (k-1)!}{k! 2^{(k-1)(d+1)}(k-1)!(i+1)!} \\ &= \frac{3^{i+2k-1}}{2^{(k-1)(d+1)}} \frac{1}{i+1} (i+k)!^d. \end{aligned} \tag{65}$$

Hence $S(u,s)$ is of Gevrey order d in the variables (u, su). Therefore the series $\psi_1(\sigma, u)$ and $\psi_2(\tau, u)$, in expression (49), are Gevrey-d in their variables. We can apply the inversion theorem in [4] to (52) to obtain that also the series φ is Gevrey-d in $(\sigma_1, \tau_1, u_\infty)$. Hence also χ_1 and χ_2 in (53) are Gevrey-d. Recall that

$$(x,y) = (x_\infty(1+\chi_1(su_\infty, tu_\infty, u_\infty)), y_\infty(1+\chi_2(su_\infty, tu_\infty, u_\infty))).$$

This ends the proof of Theorem 3. ∎

4 Proof of Theorem 4

We recall the linear map

$$L = \begin{pmatrix} q & r_0 \\ p & s_0 \end{pmatrix}$$

from Proposition 1, as well as the number

$$\alpha = \frac{\lambda(0)}{p} = -\frac{\mu(0)}{q}.$$

For $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$ we have, for $(m_1, m_2) = L^{-1}(k_1, k_2)$:

$$\begin{aligned} &\langle k, (\lambda(0), \mu(0), 0)\rangle = \alpha \langle k, (p, -q, 0)\rangle = \\ &\qquad \alpha(m_1 \langle (q,p), (p,-q)\rangle + m_2 \langle (r_0, s_0), (p, -q)\rangle) = \alpha m_2. \end{aligned} \tag{66}$$

From (66) it follows that a monomial

$$x^{k_1+1}y^{k_2}\delta^{k_3}\frac{\partial}{\partial x} \tag{67}$$

is resonant iff $m_2 = 0$, whence $(k_1, k_2, k_3) = m_1(q, p, 0) + (0, 0, k_3)$. So this monomial is of the form

$$x.x^{qm_1}y^{pm_1}\delta^{k_3}\frac{\partial}{\partial x} = x.u^{m_1}\delta^{k_3}\frac{\partial}{\partial x}.$$

In a similar way we find that the resonant monomials in the $\partial/\partial y$ component are of the form

$$y.u^{m_1}\delta^{k_3}\frac{\partial}{\partial y}.$$

From this we obtain the formal normal form in (8).

We want to apply Theorem 1 concerning the Gevrey character of the normalizing transformation. For that purpose, we compute for any nonresonant monomial (67), using (66) and the fact that $|m_2| \geq 1$:

$$|\langle (k_1+1, k_2, k_3), (\lambda(0), \mu(0), 0)\rangle - \lambda(0)| = |\alpha m_2| \geq |\alpha| > 0.$$

Similar computations for the $\partial/\partial y$ component. Hence condition (2) of that theorem is fulfilled for $s = 1$ and $K = |\alpha|$. ∎

5 Proof of Theorem 6

We want to apply Theorem 5 for a well chosen set $\mathcal{G}$. Let us define $\mathcal{G}$ as follows. For $(1, 1+k_1, k_2), (2, k_1, 1+k_2) \in \{1, 2\} \times \mathbb{N}^2_{quad}$ we consider $(m_1, m_2) = L^{-1}(k_1, k_2)$ as in Proposition 1. We say that $(1, 1+k_1, k_2)$ and $(2, k_1, 1+k_2)$ belong to $\mathcal{G}$ iff $m_2 \neq 0$. Remark that $\mathcal{G}$ corresponds to the nonresonant monomials of the form $x^{1+k_1}y^{k_2}\partial/\partial x$ and $x^{k_1}y^{k_2+1}\partial/\partial y$.

We claim that this set $\mathcal{G}$ satisfies the hypothesis of order $s = 1$ from Definition 2. Indeed, for $(1, 1+k_1, k_2) \in \mathcal{G}$ we have

$$\begin{aligned} &|\langle (\alpha(\varepsilon)(p+\varepsilon), \alpha(\varepsilon)(-q)), (1+k_1, k_2)\rangle - \alpha(\varepsilon)(p+\varepsilon)| \\ &\quad = |\alpha(\varepsilon)|.|\langle (p, -q), m_1(q, p) + m_2(r_0, s_0)\rangle + \varepsilon k_1| \\ &\quad = |\alpha(\varepsilon)|.|m_2(pr_0 - qs_0) + \varepsilon k_1| \\ &\quad = |\alpha(\varepsilon)|.|m_2 + \varepsilon k_1| \\ &\quad \geq |\alpha(\varepsilon)|.(|m_2| - |\varepsilon|.|k|). \end{aligned} \tag{68}$$

Using (22) we can continue (68), and have for all $\varepsilon \in \Lambda_{|k|}$, since $m_2 \neq 0$ is an integer:

$$\begin{aligned} |\langle (\alpha(\varepsilon)(p+\varepsilon), \alpha(\varepsilon)(-q)), (1+k_1, k_2)\rangle - \alpha(\varepsilon)(p+\varepsilon)| &\geq \frac{1}{2}|\alpha(0)|(|m_2| - \frac{1}{2}) \\ &\geq \frac{1}{4}|\alpha(0)| > 0. \end{aligned}$$

Similar estimates for $(2, k_1, 1+k_2) \in \mathcal{G}$. Hence the set $\mathcal{G}$ satisfies the hypothesis of order $s = 1$ with $K = |\alpha(0)|/4$. Therefore we can finish the proof by applying Theorem 5. ∎

6 Proof of Theorem 7

Lemma 4. *We have*

$$u' = \alpha(\varepsilon)q\varepsilon u + u(qF^1 + pG^1).$$

Proof. This is a straightforward calculation. ■

6.1 A first change of variables

We consider the new variable

$$x_1 = x - x\omega(s_\varepsilon)F^1(u, \varepsilon) \tag{69}$$

where ω is yet to be determined in such a way that terms of order not $O(u^2)$ are eliminated (precision follows below). We calculate:

$$\begin{aligned} x_1' &= \alpha(\varepsilon)(p+\varepsilon)x + xF^1 - (\alpha(\varepsilon)(p+\varepsilon)x + xF^1).\omega F^1 - x\frac{d}{dt}(\omega(s_\varepsilon))F^1 \\ &\quad - x\omega\partial_u F^1.(\alpha(\varepsilon)q\varepsilon u + u(qF^1 + pG^1)). \end{aligned}$$

We replace the first x using (69) and, for the convenience of the reader, we mark canceling terms; we also mark terms not of order $O(u^2)$:

$$\begin{aligned} x_1' &= \alpha(\varepsilon)(p+\varepsilon)(x_1 + x\omega F^1) + xF^1 - (\alpha(\varepsilon)(p+\varepsilon)x + xF^1).\omega F^1 - x\frac{d}{dt}(\omega(s_\varepsilon))F^1 \\ &\quad - x\omega\partial_u F^1.(\alpha(\varepsilon)q\varepsilon u + u(qF^1 + pG^1)) \\ &= \alpha(\varepsilon)(p+\varepsilon)x_1 + xF^1 - xF^1.\omega F^1 - x\frac{d}{dt}(\omega(s_\varepsilon))F^1 \\ &\quad - x\omega\partial_u F^1.(\alpha(\varepsilon)q\varepsilon u + u(qF^1 + pG^1)). \end{aligned} \tag{70}$$

In those terms we always have a factor x, and the remaining factor is equal to

$$F^1 - \frac{d}{dt}(\omega(s_\varepsilon))F^1 - \omega\partial_u F^1.\alpha(\varepsilon)q\varepsilon u. \tag{71}$$

Since $F^1(\varepsilon, u) = O(u)$, we have $\partial_u F^1(\varepsilon, u).u = F^1(\varepsilon, u) + O(u^2)$. If we insert this in the last term of (71) we get

$$(71) = F^1 - \frac{d}{dt}(\omega(s_\varepsilon))F^1 - \omega.(F^1 + O(u^2)).\alpha(\varepsilon)q\varepsilon.$$

We see that all the terms that are possibly not $O(u^2)$ contain the factor F^1. We want them to be zero. Therefore we want to solve

$$\begin{aligned} 0 &= 1 - \frac{d}{dt}\omega(s_\varepsilon) - \omega(s_\varepsilon)\alpha(\varepsilon)q\varepsilon \\ &\Updownarrow \\ \frac{d}{dt}\omega(s_\varepsilon) &= 1 - \omega(s_\varepsilon).\alpha(\varepsilon)q\varepsilon \\ &\Updownarrow \\ \frac{d\omega}{ds_\varepsilon}.\frac{ds_\varepsilon}{dt} &= 1 - \omega(s_\varepsilon).\alpha(\varepsilon)q\varepsilon. \end{aligned} \tag{72}$$

We can take $\omega(s_\varepsilon) = s_\varepsilon$, in view of equation (25). We insert (72) into (71):

$$(71) \quad = \quad \omega(s_\varepsilon)\alpha(\varepsilon)q\varepsilon(F^1 - \partial_u F^1.u).$$

We enter this in (70) and obtain

$$\begin{aligned} x'_1 \quad &= \quad \alpha(\varepsilon)(p+\varepsilon)x_1 - xF^1.\omega F^1 \\ &\quad + x\omega\alpha(\varepsilon)q\varepsilon(F^1 - \partial_u F^1.u) - x\omega\partial_u F^1.u(qF^1 + pG^1). \end{aligned}$$

We can rewrite this in the form

$$x'_1 = \alpha(\varepsilon)(p+\varepsilon)x_1 - x\omega F^2(\varepsilon, u)$$

where

$$F^2(\varepsilon, u) = F^1.F^1 - \alpha(\varepsilon)q\varepsilon(F^1 - \partial_u F^1.u) + \partial_u F^1.u(qF^1 + pG^1).$$

Observe that $F^2(\varepsilon, u) = O(u^2)$.

Completely similar calculations for the second component.

6.2 Induction

Assume by induction that we have, for $n \geq 2$:

$$(x'_{n-1}, y'_{n-1}) = \alpha(\varepsilon)((p+\varepsilon)x_{n-1}, -qy_{n-1}) + \frac{(-1)^{n-1}}{(n-1)!}(xs_\varepsilon^{n-1}F^n(\varepsilon, u), yt_\varepsilon^{n-1}G^n(\varepsilon, u))$$

where $(F^n, G^n)(\varepsilon, u) = O(u^n)$. We consider the new variables

$$(x_n, y_n) = (x_{n-1}, y_{n-1}) + \frac{(-1)^n}{n!}(xs_\varepsilon^n F^n(\varepsilon, u), yt_\varepsilon^n G^n(\varepsilon, u)). \tag{73}$$

With completely similar computations as in section 6.1 we obtain

$$x'_n = \alpha(\varepsilon)(p+\varepsilon).x_n + \frac{(-1)^n}{n!}xs_\varepsilon^n.F^{n+1}(\varepsilon, u)$$

where

$$F^{n+1}(\varepsilon, u) = F^1.F^n + \alpha(\varepsilon)q\varepsilon(-nF^n + \partial_u F^n.u) + \partial_u F^n.u(qF^1 + pG^1), \tag{74}$$

and

$$y'_n = -\alpha(\varepsilon)q.y_n + \frac{(-1)^n}{n!}yt_\varepsilon^n G^{n+1}(\varepsilon, u).$$

with

$$G^{n+1}(\varepsilon, u) = G^1 G^n + \alpha(\varepsilon)q\varepsilon(-nG^n + \partial_u G^n.u) + \partial_u G^n.u(qF^1 + pG^1).$$

In the same way as in section 3.2 we take the formal limit $(\tilde{x}, \tilde{y}) = \lim_{n\to\infty}(x_n, y_n)$ of (73) and shall estimate its asymptotics. We consider the first component, the second one being similar.

6.3 Asymptotics of the F^n

We recall that $F^1(\varepsilon,u) = \sum_{l\geq 1} a_l(\varepsilon)u^l$ with a_l defined on $|\varepsilon| < 1/(2dl)$ and $|a_l|_{dl} \leq Cl!^d r^l$. In the sequel, the value of $d \geq 0$ is unimportant, but as we have indicated in Theorem 6, a 'natural' choice would be $d = q + p$.

Again, just like in section 3.3, we can perform a rescaling and work with the following majorant for F^1:

$$F^1(\varepsilon,u) \preceq \sum_{n\geq 1} n!^d u^n, \tag{75}$$

and in a completely similar way we can arrange that also G^1 and $qF^1 + pG^1$ have this majorant (75).

We recall that $\eta = |\alpha(\varepsilon)q\varepsilon|$. We consider the recursion (74). Assume, by induction on N, that F^N has a majorant of the form

$$F^N(\eta,u) \preceq \sum_{n\geq N} n!^d a_{nN}(\eta)u^n.$$

with, thus,

$$a_{n1}(\eta) = 1$$

for all $n \geq 1$. We majorate F^{N+1}:

$$F^{N+1} \preceq F^1.F^N + \eta.(-NF^N + \partial_u F^N.u) + \partial_u F^N.u.(qF^1 + pG^1). \tag{76}$$

For the first term in the right hand side of (76) we have, just like in (60):

$$F^1.F^N \preceq \frac{1}{2^d} \sum_{m\geq N+1} m!^d u^m \sum_{n=N}^{m-1} a_{nN}.$$

For the second term we get

$$\begin{aligned} \eta.(-NF^N + \partial_u F^N.u) &\preceq \eta.(-N\sum_{n\geq N} n!^d a_{nN}u^n + \sum_{n\geq N} n.n!^d a_{nN}u^n) \\ &= \eta.\sum_{n\geq N+1} (n-N)n!^d a_{nN}u^n \end{aligned}$$

and for the third term we obtain

$$\begin{aligned} \partial_u F^N.u.(qF^1 + pG^1) &\preceq \sum_{n\geq N} n.n!^d a_{nN}u^n.\sum_{k\geq 1} k!^d u^k \\ &\preceq \frac{1}{2^d} \sum_{m\geq N+1} m!^d u^m \sum_{n=N}^{m-1} n a_{nN}. \end{aligned}$$

We conclude here that

$$F^{N+1} \preceq \sum_{m\geq N+1} m!^d (\frac{1}{2^d} \sum_{n=N}^{m-1} (n+1)a_{nN} + \eta.(m-N)a_{mN})u^m.$$

It follows that

$$a_{m,N+1}(\eta) = \frac{1}{2^d}\sum_{n=N}^{m-1}(n+1)a_{nN}(\eta) + \eta.(m-N)a_{mN}(\eta).$$

Now we proceed using induction and propose the estimate

$$\begin{aligned} a_{m,N+1}(\eta) \;\leq\; & \frac{1}{2^{N(d+1)}N!}(m+N)(m+N-1)\dots(m-N+1) \\ & + \sum_{j=1}^{N-1}\eta^j\beta_{jN}(m+N-j)\dots(m-N+1) \\ & + \eta^N(m-1)(m-2)\dots(m-N) \qquad (77) \end{aligned}$$

for β_{jN} to be determined. Induction gives a majorant

$$\begin{aligned} a_{m,N+2} \;\leq\; & \frac{1}{2^d}\sum_{n=N+1}^{m-1}(n+1)a_{n,N+1}(\eta) + \eta.(m-N-1)a_{m,N+1}(\eta) \\ \leq\; & \frac{1}{2^d}\sum_{n=N+1}^{m-1}(n+1)[\frac{1}{2^{N(d+1)}N!}(n+N)(n+N-1)\dots(n-N+1) \qquad (78) \\ & + \sum_{j=1}^{N-1}\eta^j\beta_{jN}(n+N-j)\dots(n-N+1) \qquad (79) \\ & + \eta^N(n-1)(n-2)\dots(n-N)] \qquad (80) \\ & + \eta.(m-N-1)[\frac{1}{2^{N(d+1)}N!}(m+N)(m+N-1)\dots(m-N+1) \qquad (81) \\ & + \sum_{j=1}^{N-1}\eta^j\beta_{jN}(m+N-j)\dots(m-N+1) \qquad (82) \\ & + \eta^N(m-1)(m-2)\dots(m-N)] \qquad (83) \end{aligned}$$

in which we will treat now the terms (78) to (83) separately. For convenience we recall the formula

$$\sum_{n=K}^{P}(n-1)(n-2)\dots(n-K+1) = \frac{1}{K}P.(P-1)\dots(P-K+1). \qquad (84)$$

First, in precisely the same way as in (64):

$$\begin{aligned} (78) \;\leq\; & \frac{1}{2^d}\sum_{n=N+1}^{m-1}(n+N+1)\frac{1}{2^{N(d+1)}N!}(n+N)(n+N-1)\dots(n-N+1) \\ =\; & \frac{1}{2^{(N+1)(d+1)}}\frac{1}{(N+1)!}(m+N+1)(m+N)\dots(m-N). \end{aligned}$$

Second:

$$\begin{aligned}
(79) &= \frac{1}{2^d}\sum_{n=N+1}^{m-1}(n+1)\sum_{j=1}^{N-1}\eta^j\beta_{jN}(n+N-j)\dots(n-N+1)\\
&\le \frac{1}{2^d}\sum_{j=1}^{N-1}\eta^j\beta_{jN}\sum_{n=N+1}^{m-1}(n+N-j+1)(n+N-j)\dots(n-N+1)\\
&\le \frac{1}{2^d}\sum_{j=1}^{N-1}\eta^j\beta_{jN}\sum_{n=2N+2-j}^{m+N-j+1}(n-1)\dots(n-(2N+1-j))\\
&\quad \text{(apply formula (84) for } K=2N+2-j \text{ and } P=m+N-j+1)\\
&= \frac{1}{2^d}\sum_{j=1}^{N-1}\eta^j\beta_{jN}\frac{1}{2N+2-j}(m+N-j+1)\dots(m-N).
\end{aligned}$$

Third:

$$\begin{aligned}
(80) &= \frac{1}{2^d}\sum_{n=N+1}^{m-1}(n+1).\eta^N(n-1)(n-2)\dots(n-N)\\
&\le \frac{1}{2^d}\eta^N\sum_{n=N+1}^{m-1}(n+1).n(n-1)\dots(n-N+1)\\
&\le \frac{1}{2^d}\eta^N\sum_{n=N+2}^{m+1}(n-1)(n-2)\dots(n-N-1)\\
&\quad \text{(apply formula (84) for } K=N+2 \text{ and } P=m+1)\\
&= \frac{1}{2^d}\eta^N\frac{1}{N+2}(m+1)m\dots(m-N).
\end{aligned}$$

Fourth:

$$\begin{aligned}
(81) &= \eta.(m-N-1)\frac{1}{2^{N(d+1)}N!}(m+N)(m+N-1)\dots(m-N+1)\\
&\le \eta.\frac{1}{2^{N(d+1)}N!}(m+N)(m+N-1)\dots(m-N+1).(m-N).
\end{aligned}$$

Fifth:

$$\begin{aligned}
(82) &= \eta.(m-N-1).\sum_{j=1}^{N-1}\eta^j\beta_{jN}(m+N-j)\dots(m-N+1)\\
&= \sum_{j=2}^{N}\eta^j\beta_{j-1,N}(m+N-j+1)\dots(m-N+1).(m-N-1)\\
&\le \sum_{j=2}^{N}\eta^j\beta_{j-1,N}(m+N-j+1)\dots(m-N+1).(m-N).
\end{aligned}$$

Sixth:

$$(83) = \eta^{N+1}(m-1)\dots(m-N).(m-N-1).$$

We derive the following recursion for the β_{jN} now. Since, for $N = 1$, we have for all $m \geq 2$:

$$a_{m2} \leq \frac{1}{2^{d+1}}(m+1)m + \eta(m-1), \tag{85}$$

we can calculate for all $m \geq 3$

$$\begin{aligned}
a_{m3} &= \frac{1}{2^d}\sum_{n=2}^{m-1}(n+1)a_{n2}(\eta) + \eta.(m-2)a_{m2}(\eta) \\
&\leq \frac{1}{2^d}\sum_{n=2}^{m-1}(n+1)[\frac{1}{2^{d+1}}(n+1)n + \eta(n-1)] \\
&\quad + \eta.(m-2)[\frac{1}{2^{d+1}}(m+1)m + \eta(m-1)] \\
&\leq \frac{1}{2^d}\sum_{n=2}^{m-1}(n+2)\frac{1}{2^{d+1}}(n+1)n + \eta\frac{1}{2^d}\sum_{n=2}^{m-1}(n+1)n + \eta.(m-2)\frac{1}{2^{d+1}}(m+1)m \\
&\quad + \eta^2(m-2)(m-1) \\
&\leq \frac{1}{2^{(d+1)+d}}\sum_{n=4}^{m+2}(n-1)(n-2)(n-3) + \eta.\frac{1}{2^d}\sum_{n=3}^{m+1}(n-1)(n-2) + \\
&\quad \eta.(m-2)\frac{1}{2^{d+1}}(m+1)m + \eta^2(m-2)(m-1) \\
&= \frac{1}{2^{(d+1)+d}}.\frac{1}{4}(m+2)(m+1)m(m-1) + \eta.\frac{1}{2^d}\frac{1}{3}(m+1)m(m-1) + \\
&\quad \eta.(m-2)\frac{1}{2^{d+1}}(m+1)m + \eta^2(m-2)(m-1) \\
&\leq \frac{1}{2^{2(d+1)}}.\frac{1}{4}(m+2)(m+1)m(m-1) + \eta.(\frac{1}{2^d}\frac{1}{3} + \frac{1}{2^{d+1}}).(m+1)m(m-1) \\
&\quad + \eta^2(m-2)(m-1).
\end{aligned}$$

We see that

$$\beta_{12} = \frac{1}{2^d}\frac{1}{3} + \frac{1}{2^{d+1}}. \tag{86}$$

Let $N \geq 2$. We want

$$\begin{aligned}
a_{m,N+2} &\leq \frac{1}{2^{(N+1)(d+1)}}(m+N+1)\dots(m-N) + \\
&\sum_{j=1}^{N}\eta^j\beta_{j,N+1}(m+N-j+1)\dots(m-N) + \eta^{N+1}(m-1)\dots(m-N-1).
\end{aligned} \tag{87}$$

The first and third term above correspond to (78) and (83). For the second term in (87) we infer the following recursion. For $j = 1$ we have

$$\beta_{1,N+1} = \frac{1}{2^d}\beta_{1N}\frac{1}{2N+1} + \frac{1}{2^{N(d+1)}N!}; \tag{88}$$

for $2 \leq j \leq N-1$:

$$\beta_{j,N+1} = \frac{1}{2^d}\beta_{jN}\frac{1}{2N+2-j} + \beta_{j-1,N}; \tag{89}$$

for $j = N$:

$$\beta_{N,N+1} = \frac{1}{2^d}\frac{1}{N+2} + \beta_{N-1,N}. \tag{90}$$

Lemma 5. *For all $N \geq 2$ we have the estimates*

$$\beta_{1N} \leq \frac{1}{2^{(N-1)(d+1)}(N-1)!}\binom{N}{1}, \tag{91}$$

$$\beta_{jN} \leq \frac{1}{2^{(N-j)(d+1)}}\binom{N}{j} \text{ for } 2 \leq j \leq N-2, \tag{92}$$

$$\beta_{N-1,N} \leq \frac{1}{2^{d+1}}\binom{N}{N-1}. \tag{93}$$

Proof. For $N = 2$ this follows from (86); note that (91) and (93) are the same in this case. Assume, by induction, that the lemma is true for $N \geq 2$. Below, we shall apply Pascal's triangle identity several times. First of all, we use (88):

$$\begin{aligned}
\beta_{1,N+1} &= \frac{1}{2^d}\beta_{1N}\frac{1}{2N+1} + \frac{1}{2^{N(d+1)}N!} \\
&= \frac{1}{2^d 2^{(N-1)(d+1)}(N-1)!}\binom{N}{1}\frac{1}{2N+1} + \frac{1}{2^{N(d+1)}N!}\binom{N}{0} \\
&\leq \frac{1}{2^d 2^{(N-1)(d+1)}(N-1)!}\frac{1}{2N}\binom{N}{1} + \frac{1}{2^{N(d+1)}N!}\binom{N}{0} \\
&= \frac{1}{2^{N(d+1)}N!}(\binom{N}{1} + \binom{N}{0}) \\
&= \frac{1}{2^{N(d+1)}N!}\binom{N+1}{1}.
\end{aligned}$$

Second, for $2 \leq j \leq N-1$ we apply (89):

$$\begin{aligned}
\beta_{j,N+1} &= \frac{1}{2^d}\beta_{jN}\frac{1}{2N+2-j} + \beta_{j-1,N} \\
&\leq \frac{1}{2^d 2^{(N-j)(d+1)}(N+3)}\binom{N}{j} + \frac{1}{2^{(N-j+1)(d+1)}}\binom{N}{j-1} \\
&\leq \frac{1}{2^d 2^{(N-j)(d+1)}.2}\binom{N}{j} + \frac{1}{2^{(N-j+1)(d+1)}}\binom{N}{j-1} \\
&= \frac{1}{2^{(N-j+1)(d+1)}}(\binom{N}{j} + \binom{N}{j-1}) \\
&= \frac{1}{2^{(N-j+1)(d+1)}}\binom{N+1}{j}.
\end{aligned}$$

Finally, we use (90):

$$\begin{aligned}\beta_{N,N+1} &= \frac{1}{2^d}\frac{1}{N+2} + \beta_{N-1,N}\\ &= \frac{1}{2^d}\frac{1}{N+2}\binom{N}{N} + \frac{1}{2^{d+1}}\binom{N}{N-1}\\ &\leq \frac{1}{2^{d+1}}(\binom{N}{N} + \binom{N}{N-1})\\ &= \frac{1}{2^{d+1}}\binom{N+1}{N}.\end{aligned}$$

The induction is finished. ∎

From (77), (85) and Lemma 5 we obtain:

Lemma 6. *The following estimates hold:*

$$a_{m2}(\eta) \leq \frac{1}{2^{d+1}}(m+1)m + \eta(m-1)$$

and for all $N \geq 2$:

$$a_{m,N+1}(\eta) \leq \frac{1}{2^N(d+1)N!}\frac{(m+N)!}{(m-N)!} + \sum_{j=1}^{N-1}\eta^j\frac{1}{2^{(N-j)(d+1)}}\binom{N}{j}\frac{(m+N-j)!}{(m-N)!} + \eta^N\frac{(m-1)!}{(m-N-1)!}.$$

Lemma 7. *We have the following majorants:*

$$F^2(\eta,u) \preceq \sum_{n\geq 2} n!^d(\frac{1}{2^{d+1}}(n+1)n + \eta(n-1))u^n$$

and for all $N \geq 3$:

$$\begin{aligned}F^N(\eta,u) \preceq \sum_{n\geq N} n!^d(&\frac{1}{2^{(N-1)(d+1)}(N-1)!}\frac{(n+N-1)!}{(n-N+1)!}\\ &+\sum_{j=1}^{N-2}\eta^j\frac{1}{2^{(N-1-j)(d+1)}}\binom{N-1}{j}\frac{(n+N-1-j)!}{(n-N+1)!}\\ &+\eta^{N-1}\frac{(n-1)!}{(n-N)!})u^n.\end{aligned}$$

6.4 Asymptotics of the 'final' transformation

Using Lemma 7 we have, for the left hand side in (30), a majorant of the form

$$S(\varepsilon,u,s) = s_\varepsilon\sum_{n\geq 1} n!^d u^n + \frac{s_\varepsilon^2}{2!}\sum_{n\geq 2} n!^d(\frac{1}{2^{d+1}}(n+1)n + \eta(n-1))u^n + \sum_{k\geq 3}\sum_{n\geq k}\sum_{j=0}^{k-1}\gamma_{nkj}\eta^j u^n s_\varepsilon^k \quad (94)$$

where, as before, $\eta = |\alpha(\varepsilon)q\varepsilon|$ and where the coefficients γ_{nkj} are:

$$\begin{aligned}\gamma_{nk0} &= \frac{n!^d(n+k-1)!}{k!2^{(k-1)(d+1)}(k-1)!(n-k+1)!},\\ \gamma_{nkj} &= \frac{n!^d}{k!2^{(k-1-j)(d+1)}}\binom{k-1}{j}\frac{(n+k-1-j)!}{(n-k+1)!} \text{ for } 1\le j\le k-2,\\ \gamma_{nk,k-1} &= \frac{n!^d(n-1)!}{k!(n-k)!}.\end{aligned}$$

We rewrite the summation in (94) as follows:

$$\sum_{k\ge 3}\sum_{n\ge k}\sum_{j=0}^{k-1}\gamma_{nkj}\eta^j u^n s_\varepsilon^k = \sum_{i\ge 0}u^i\sum_{k\ge 3}(s_\varepsilon u)^k\sum_{j=0}^{k-1}\gamma_{i+k,kj}\eta^j.$$

Let us now examine and estimate the coefficients $f_{ikj} = \gamma_{i+k,k,j}$. For $j=0$ we have, in precisely the same way as before in (65):

$$f_{ik0} = \gamma_{i+k,k,0} \le \frac{3^{i+2k-1}}{2^{(k-1)(d+1)}}\frac{1}{i+1}(i+k)!^d.$$

For $1\le j\le k-2$ we can make the estimates

$$\begin{aligned}f_{ikj} = \gamma_{i+k,k,j} &= \frac{(i+k)!^d}{k!2^{(k-1-j)(d+1)}}\binom{k-1}{j}\frac{(i+2k-1-j)!}{(i+1)!}\\ &\le \frac{(i+k)!^d}{k!2^{(k-1-j)(d+1)}}\binom{k-1}{j}3^{i+2k-1}\frac{i!k!(k-1)!}{(i+1)!}\\ &\le (i+k)!^d 2^{-kd+d+jd+j}3^{i+2k-1}\frac{(k-1)!}{i+1},\end{aligned}$$

and finally for $j=k-1$:

$$\begin{aligned}f_{ik,k-1} = \gamma_{i+k,k,k-1} &= \frac{(i+k)!^d(i+k-1)!}{k!i!}\\ &\le \frac{(i+k)!^d 2^{i+k-1}i!(k-1)!}{k!i!}\\ &= (i+k)!^d 2^{i+k-1}\frac{1}{k}.\end{aligned}$$

From these estimates it follows that the series in (30) is of Gevrey order $d+1$ in the variables $(u, s_\varepsilon u)$. We can proceed in the same way as in section 3. ∎

Remark 9. From the above calculations it is unclear to us whether these Gevrey type results can be sharpened or not, in particular the estimates leading to the terms with $1\le j\le N-2$ in Lemma 7.

7 Proof of Theorem 8

We recall the linear maps L and M from Proposition 1 which are one-to-one from $\mathbb{Z}^2$ to $\mathbb{Z}^2$. We define the sets

$$\tilde{B}_1 = \{(m_1, m_2) \in \mathbb{Z}^2 | m_2 \geq 0 \text{ and } m_1 \geq K m_2 + 1\},$$

$$\tilde{G}_1 = \{(m_1, m_2) \in \mathbb{Z}^2 | m_2 \geq 0 \text{ and } \frac{-s_0}{p} m_2 \leq m_1 \leq K m_2\},$$

$B_1 = L(\tilde{B}_1)$ and $G_1 = L(\tilde{G}_1)$. Similarly:

$$\tilde{B}_2 = \{(m_1, m_2) \in \mathbb{Z}^2 | m_2 \geq 1 \text{ and } m_1 \geq K m_2 + 1\}, \quad (95)$$

$$\tilde{G}_2 = \{(m_1, m_2) \in \mathbb{Z}^2 | m_2 \geq 0 \text{ and } \frac{-s_1}{p} m_2 \leq m_1 \leq K m_2\},$$

$B_2 = M(\tilde{B}_2)$ and $G_2 = M(\tilde{G}_2)$.

Remark 10. Notice that the sets B_1 and B_2 do not overlap, because of the following. The matrix

$$L - M = \begin{pmatrix} 0 & r_0 - r_1 \\ 0 & s_0 - s_1 \end{pmatrix}$$

cannot be zero, since $(r_0 - r_1, s_0 - s_1) = (r_0 - q + r_0, s_0 - p + s_0) = 2(r_0, s_0) - (q, p) \neq (0, 0)$ because $\gcd(q, p) = 1$. Hence the kernel of $L - M$ is equal to $\{m_2 = 0\}$, and we have removed $m_2 = 0$ from the definition of $\tilde{B}_2$ in (95).

We could call, informally, the sets G_1 and G_2 'good', because we will explain below why monomials $x^{k_1} y^{k_2}$ in the series (32) of X_ε with (k_1, k_2) in a good set can be removed by an analytic change of variables. The sets B_1 and B_2 could be called 'bad'.

We have, for all $(k_1, k_2) \in \mathbb{Z}^2$, writing $(m_1, m_2) = L^{-1}(k_1, k_2)$:

$$\begin{aligned} \langle (\alpha(\varepsilon)(p + \varepsilon, -q), (k_1, k_2) \rangle &= \langle (\alpha(\varepsilon)(p + \varepsilon, -q), m_1(q, p) + m_2(r_0, s_0) \rangle \\ &= \alpha(\varepsilon)(m_1 \varepsilon q + m_2(1 + \varepsilon r_0)). \end{aligned}$$

Let $(k_1, k_2) \in G_1$. Then $|m_1| \leq \max\{\frac{s_0}{p}, K\} m_2$ and hence, on the one hand:

$$\begin{aligned} |\langle (\alpha(\varepsilon)(p + \varepsilon, -q), (k_1, k_2) \rangle| &\geq |\alpha(\varepsilon)|(|m_2(1 + \varepsilon r_0)| - |m_1||\varepsilon| q) \\ &\geq |\alpha(\varepsilon)| m_2(|1 + \varepsilon r_0| - \max\{\frac{s_0}{p}, K\}|\varepsilon| q); \end{aligned}$$

on the other hand:

$$\begin{aligned} |(k_1, k_2)| &\leq |m_1|(q + p) + m_2(r_0 + s_0) \\ &\leq m_2(\max\{\frac{s_0}{p}, K\}(q + p) + r_0 + s_0). \end{aligned}$$

We obtain:

$$|\langle (\alpha(\varepsilon)(p + \varepsilon, -q), (k_1, k_2) \rangle| \geq \frac{|\alpha(\varepsilon)|(|1 + \varepsilon r_0| - \max\{\frac{s_0}{p}, K\}|\varepsilon| q)}{\max\{\frac{s_0}{p}, K\}(q + p) + r_0 + s_0} |(k_1, k_2)|. \quad (96)$$

Since $\alpha(0) \neq 0$ and by continuity, there exists $\varepsilon_0 > 0$ such that, for the numerator in (96):

$$\min_{|\varepsilon|\leq\varepsilon_0} |\alpha(\varepsilon)|(|1+\varepsilon r_0| - \max\{\frac{s_0}{p}, K\}|\varepsilon|q) > 0.$$

Hence, the hypothesis (2) of Theorem 1 is fulfilled for the set G_1, for $s = 0$. In a completely analogous way we obtain a similar condition for the set G_2. We infer that there exists an analytic transformation ψ_ε of the form (33) conjugating X_ε into $Y_\varepsilon = \psi_\varepsilon^* X_\varepsilon$ of the following form (let us rename $m_2 = k$ in order to get the expression in (34)):

$$\begin{aligned}
Y_\varepsilon(x,y) &= \alpha(\varepsilon).((p+\varepsilon)x\frac{\partial}{\partial x} - qy\frac{\partial}{\partial y}) \\
&+(\sum_{k\geq 0} u^{Kk} \sum_{m_1\geq Kk+1} b_{m_1 k}(\varepsilon)u^{m_1-Kk}(x^{r_0}y^{s_0})^k \\
&+\sum_{k\geq 1} u^{Kk} \sum_{m_1\geq Kk+1} \tilde{b}_{m_1 k}(\varepsilon)u^{m_1-Kk}(x^{r_1}y^{s_1})^k)x\frac{\partial}{\partial x} \\
&+(\sum_{k\geq 0} u^{Kk} \sum_{m_1\geq Kk+1} c_{m_1 k}(\varepsilon)u^{m_1-Kk}(x^{r_0}y^{s_0})^k \\
&+\sum_{k\geq 1} u^{Kk} \sum_{m_1\geq Kk+1} \tilde{c}_{m_1 k}(\varepsilon)u^{m_1-Kk}(x^{r_1}y^{s_1})^k)y\frac{\partial}{\partial y}.
\end{aligned}$$

In both ψ_ε and Y_ε the coefficient functions of the series belong to the algebra $\mathcal{A}(\varepsilon_0)$, provided thus that ε_0 is sufficiently small. Now it suffices to define

$$\begin{aligned}
F_k(\varepsilon,u) &= \sum_{m_1\geq Kk+1} b_{m_1 k}(\varepsilon)u^{m_1-Kk}, \\
\tilde{F}_k(\varepsilon,u) &= \sum_{m_1\geq Kk+1} \tilde{b}_{m_1 k}(\varepsilon)u^{m_1-Kk}, \\
G_k(\varepsilon,u) &= \sum_{m_1\geq Kk+1} c_{m_1 k}(\varepsilon)u^{m_1-Kk}, \\
\tilde{G}_k(\varepsilon,u) &= \sum_{m_1\geq Kk+1} \tilde{c}_{m_1 k}(\varepsilon)u^{m_1-Kk}
\end{aligned}$$

in order to obtain the expression (34) for Y_ε in the statement of the theorem. Observe that $F_k(\varepsilon,0) = \tilde{F}_k(\varepsilon,0) = G_k(\varepsilon,0) = \tilde{G}_k(\varepsilon,0) = 0$. ∎

8 Proof of Theorem 9

We recall that $u = x^q y^p$ and that v_k is defined as in (36).We will need to compute u' and v_k' below, and define for that purpose

$$\Sigma_\varepsilon(k) = \begin{cases} k\alpha(\varepsilon)(1+r_0\varepsilon) \text{ if } k\geq 0 \\ (-k)\alpha(\varepsilon)(-1+r_1\varepsilon) \text{ if } k<0 \end{cases} \quad (97)$$

and also

$$H_k^{k_2} = \begin{cases} k(r_0F_{k_2} + s_0G_{k_2}) \text{ if } k\geq 0 \\ -k(r_1\tilde{F}_{k_2} + s_1\tilde{G}_{k_2}) \text{ if } k<0. \end{cases}$$

From a straightforward computation one obtains:

Lemma 8. *The following formulas hold:*

$$u' = \varepsilon\alpha(\varepsilon)qu + u\sum_k v_k(qF_k(\varepsilon,u) + pG_k(\varepsilon,u))$$

and

$$v_k' = \Sigma_\varepsilon(k)v_k + v_k\sum_{k_2} v_{k_2}H_k^{k_2}(\varepsilon,u). \tag{98}$$

8.1 A first change of variables

We consider the new variable

$$x_1 = x - x\sum_k \omega_k(s_\varepsilon)v_kF_k(\varepsilon,u)$$

where ω_k is yet to be determined in such a way that terms of order not $O(u^2)$ are eliminated (precision follows below). We calculate, in a similar way as section 6.1 (but slightly more involved), marking $\underline{\text{terms not of order } O(u^2)}$:

$$\begin{aligned}
x_1' \quad = \quad & \alpha(\varepsilon)(p+\varepsilon)x_1 + \underline{x\sum_k v_kF_k} \\
& -x\sum_{k_2} v_{k_2}F_{k_2}.\sum_k \omega_kv_kF_k - \underline{x\sum_k \frac{d}{dt}(\omega_k(s_\varepsilon)).v_kF_k} \\
& -x\sum_k \omega_k.(\underline{\Sigma_\varepsilon(k)v_k} + v_k\sum_{k_2} v_{k_2}H_k^{k_2})F_k \\
& -x\sum_k \omega_k.v_k\partial_uF_k.(\underline{\alpha(\varepsilon)q\varepsilon u} + u\sum_{k_2} v_{k_2}(qF_{k_2} + pG_{k_2})).
\end{aligned} \tag{99}$$

In $\underline{\text{those terms}}$ we always have a factor $x.v_k$, and the remaining factor is equal to

$$F_k - \frac{d}{dt}\omega_k(s_\varepsilon).F_k - \omega_k(s_\varepsilon)\Sigma_\varepsilon(k).F_k - \omega_k(s_\varepsilon).\partial_uF_k.\alpha(\varepsilon)q\varepsilon u. \tag{100}$$

Since $F_k(\varepsilon,u) = O(u)$ we have $\partial_uF_k(\varepsilon,u).u = F_k(\varepsilon,u) + O(u^2)$, and if we insert this in the last term of (100) we get that this is equal to

$$F_k - \frac{d}{dt}\omega_k(s_\varepsilon).F_k - \omega_k(s_\varepsilon)\Sigma_\varepsilon(k).F_k - \omega_k(s_\varepsilon).(F_k + O(u^2))\alpha(\varepsilon)q\varepsilon.$$

We observe that all the terms that are not $O(u^2)$ contain the factor F_k. We want them to be zero. This comes down to solving the equation

$$\begin{aligned}
\frac{d}{dt}\omega_k(s_\varepsilon) \quad &= \quad 1 - \omega_k(s_\varepsilon)(\Sigma_\varepsilon(k) + \alpha(\varepsilon)q\varepsilon) && (101) \\
&\Updownarrow \\
\frac{d\omega_k}{ds_\varepsilon}.\frac{ds_\varepsilon}{dt} \quad &= \quad 1 - \omega_k(s_\varepsilon)(\Sigma_\varepsilon(k) + \alpha(\varepsilon)q\varepsilon). && (102)
\end{aligned}$$

In the case that $k \neq 0$, we have $\Sigma_0(k) = k\alpha(0) \neq 0$ and hence we can take

$$\omega_k(s_\varepsilon) = \frac{1}{\Sigma_\varepsilon(k) + \alpha(\varepsilon)q\varepsilon} \tag{103}$$

provided that the denominator is nonzero. In the case that $k = 0$, we take

$$\omega_0(s_\varepsilon) = s_\varepsilon \tag{104}$$

and check indeed that the left hand side of (102) is equal to, using (25):

$$\frac{ds_\varepsilon}{dt} = 1 - \alpha(\varepsilon)q\varepsilon s_\varepsilon$$

and the right hand side of (102) is equal to

$$1 - \omega_0(s_\varepsilon)(\Sigma_\varepsilon(0) + \alpha(\varepsilon)q\varepsilon) = 1 - s_\varepsilon\alpha(\varepsilon)q\varepsilon.$$

We insert (101) into (100), and this respectively into (99), and obtain

$$\begin{aligned} x_1' &= \alpha(\varepsilon)(p+\varepsilon)x_1 - x\sum_{k_2} v_{k_2}F_{k_2}.\sum_k \omega_k v_k F_k - x\sum_{k\geq 0}\omega_k.v_k\sum_{k_2} v_{k_2}H_k^{k_2}F_k \\ &\quad + x\sum_k \omega_k.v_k.\alpha(\varepsilon)q\varepsilon(F_k - \partial_u F_k.u) - x\sum_k \omega_k v_k \partial_u F_k.u\sum_{k_2} v_{k_2}(qF_{k_2} + pG_{k_2}). \end{aligned}$$

We can rewrite this in the form

$$x_1' = \alpha(\varepsilon)(p+\varepsilon)x_1 - x\sum_k \omega_k v_k \sum_{k_2} v_{k_2} F_{(k,k_2)}(\varepsilon, u)$$

where $F_{(k,k_2)}$ is defined as follows:
1. for $k_2 = 0$ we take

$$F_{(k,0)} = F_0F_k + H_k^0.F_k - \alpha(\varepsilon)q\varepsilon(F_k - \partial_u F_k.u) + \partial_u F_k.u(qF_0 + pG_0);$$

2. for $k_2 \neq 0$ we take

$$F_{(k,k_2)} = F_{k_2}F_k + H_k^{k_2}.F_k + \partial_u F_k.u(qF_{k_2} + pG_{k_2}).$$

The second component is treated in the same way.

8.2 Induction

Assume by induction that we have, for $n \geq 2$:

$$\begin{cases} x_{n-1}' = \alpha(\varepsilon)(p+\varepsilon)x_{n-1} + (-1)^{n-1}x\sum_{K_n}\omega_{K_{n-1}}(s_\varepsilon)v_{K_n}F_{K_n}(\varepsilon,u) \\ y_{n-1}' = -\alpha(\varepsilon)qy_{n-1} + (-1)^{n-1}y\sum_{K_n}\omega_{K_{n-1}}(s_\varepsilon)v_{K_n}G_{K_n}(\varepsilon,u) \end{cases}$$

with $(F_{K_n}, G_{K_n})(\varepsilon, u) = O(u^n)$. In order to compute v_{K_n}' we define the quantities, by induction from (97), as

$$\Sigma_\varepsilon(K_n) = \Sigma_\varepsilon(K_{n-1}) + \Sigma_\varepsilon(k_n).$$

We already had in (98):

$$v_k' = \Sigma_\varepsilon(k)v_k + v_k \sum_{k_2} v_{k_2} H_k^{k_2}(\varepsilon, u).$$

By induction one readily shows that

$$v_{K_n}' = \Sigma_\varepsilon(K_n)v_{K_n} + v_{K_n}.\sum_k v_k H_{K_n}^k(\varepsilon, u)$$

where

$$H_{K_n}^k(\varepsilon, u) \quad = \quad H_{K_{n-1}}^k(\varepsilon, u) + H_{k_n}^k(\varepsilon, u).$$

We define the new variables

$$\begin{cases} x_n = x_{n-1} + (-1)^n x \sum_{K_n} \omega_{K_n}(s_\varepsilon) v_{K_n} F_{K_n}(\varepsilon, u) \\ y_n = y_{n-1} + (-1)^n y \sum_{K_n} \omega_{K_n}(s_\varepsilon) v_{K_n} G_{K_n}(\varepsilon, u) \end{cases}$$

where we choose $\omega_{K_n}(s_\varepsilon)$ to be a solution of the differential equation

$$-\omega_{K_{n-1}} + \frac{d}{dt}\omega_{K_n} + \omega_{K_n}\Sigma_\varepsilon(K_n) + \omega_{K_n}.\alpha(\varepsilon)q\varepsilon.n = 0$$

or equivalently, using equation (25):

$$\frac{d\omega_{K_n}}{ds_\varepsilon}.\frac{ds_\varepsilon}{dt} = \frac{d\omega_{K_n}}{ds_\varepsilon}.(1 - \alpha(\varepsilon)q\varepsilon s_\varepsilon) = \omega_{K_{n-1}} - \omega_{K_n}.(\Sigma_\varepsilon(K_n) + n\alpha(\varepsilon)q\varepsilon). \tag{105}$$

Theoretically this has, of course, a solution, but we will come back to this more explicitly below. A straightforward computation, comparable to the one in section 6.1, shows that, by this choice of ω_{K_n}, the terms not of order $O(u^{n+1})$ are eliminated. Moreover, if we denote $K_{n+1} = (K_n, k_{n+1})$, then we find the following equations for (x_n, y_n):

$$\begin{cases} x_n' = \alpha(\varepsilon)(p+\varepsilon)x_n + (-1)^n x \sum_{K_{n+1}} \omega_{K_n}(s_\varepsilon) v_{K_{n+1}} F_{K_{n+1}}(\varepsilon, u) \\ y_n' = -\alpha(\varepsilon)q y_n + (-1)^n y \sum_{K_{n+1}} \omega_{K_n}(s_\varepsilon) v_{K_{n+1}} G_{K_{n+1}}(\varepsilon, u) \end{cases}$$

where $(F_{K_{n+1}}, G_{K_{n+1}})$ is defined as follows:
1. for $k_{n+1} = 0$ we take

$$F_{(K_n,0)} = (F_0 + H_{K_n}^0)G_{K_n} + \alpha(\varepsilon)q\varepsilon(\partial_u F_{K_n}.u - nF_{K_n}) + \partial_u F_{K_n}.u(qF_0 + pG_0), \tag{106}$$

$$G_{(K_n,0)} = (G_0 + H_{K_n}^0)G_{K_n} + \alpha(\varepsilon)q\varepsilon(\partial_u G_{K_n}.u - nG_{K_n}) + \partial_u G_{K_n}.u(qF_0 + pG_0); \tag{107}$$

2. for $k_{n+1} \neq 0$ we take

$$F_{K_{n+1}} = (F_{k_{n+1}} + H_{K_n}^{k_{n+1}})F_{K_n} + \partial_u F_{K_n}.u(qF_{k_{n+1}} + pG_{k_{n+1}}), \tag{108}$$

$$G_{K_{n+1}} = (G_{k_{n+1}} + H_{K_n}^{k_{n+1}})G_{K_n} + \partial_u G_{K_n}.u(qF_{k_{n+1}} + pG_{k_{n+1}}). \tag{109}$$

Observe that $(F_{K_{n+1}}, G_{K_{n+1}}) = O(u^{n+1})$.

8.3 On the sequence of differential equations defining the ω_{K_n}

We had defined the functions ω_k, $k \in \mathbb{Z}$, in (103) and (104), and by induction we had the differential equations (105) for the ω_{K_n}, that is:

$$\frac{d\omega_{K_n}}{ds_\varepsilon}.(1-\alpha(\varepsilon)q\varepsilon s_\varepsilon) = \omega_{K_{n-1}} - \omega_{K_n}.(\Sigma_\varepsilon(K_n) + n\alpha(\varepsilon)q\varepsilon) \tag{110}$$

for $K_n \in \mathbb{Z}^n$, $n \geq 2$. Let us come back to the numbers $\Sigma_\varepsilon(K_n)$. For $n = 1$ we recall the formulas in (97) and also the fact that $r_1 = q - r_0$. Hence we can write for $k < 0$:

$$\Sigma_\varepsilon(k) = k\alpha(\varepsilon)(1 - r_1\varepsilon) = k\alpha(\varepsilon)(1 + r_0)\varepsilon + |k|.\alpha(\varepsilon)q\varepsilon.$$

Summarizing, we can write for all $k \in \mathbb{Z}$:

$$\Sigma_\varepsilon(k) = k\alpha(\varepsilon)(1 + r_0\varepsilon) + \delta_k|k|\alpha(\varepsilon)q\varepsilon$$

where δ_k is defined by (39) in Notation 3. For $n \geq 1$ and $K_n = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ we denote

$$\mathcal{S}(K_n) \;=\; \sum_{i=1}^{n} k_i.$$

By induction one easily shows:

$$\Sigma_\varepsilon(K_n) = \mathcal{S}(K_n)\alpha(\varepsilon)(1 + r_0\varepsilon) - \langle \delta_{K_n}, K_n\rangle \alpha(\varepsilon)q\varepsilon. \tag{111}$$

Observe that the factor $-\langle \delta_{K_n}, K_n\rangle$ is always a nonnegative integer. Let us fix ε for a moment and denote $s_\varepsilon = s$. Although this is not essential, we can avoid some hassle with the minus signs by considering the variable $\sigma = -s$ and by studying instead the functions

$$\tau_{K_n}(\sigma) = (-1)^n \omega_{K_n}(-\sigma). \tag{112}$$

We have, for $k \neq 0$:

$$\tau_k(\sigma) = -\frac{1}{\Sigma_\varepsilon(k) + \alpha(\varepsilon)q\varepsilon}$$

and

$$\tau_0(\sigma) = \sigma.$$

The differential equation for τ_{K_n} is then:

$$\frac{d\tau_{K_n}}{d\sigma}.(1 + \alpha(\varepsilon)q\varepsilon\sigma) = \tau_{K_{n-1}} + \tau_{K_n}.(\Sigma_\varepsilon(K_n) + n\alpha(\varepsilon)q\varepsilon),$$

and using (111) this becomes

$$\frac{d\tau_{K_n}}{d\sigma}.(1 + \alpha(\varepsilon)q\varepsilon\sigma) = \tau_{K_{n-1}} + \tau_{K_n}.(\mathcal{S}(K_n)\alpha(\varepsilon)(1 + r_0\varepsilon) + (-\langle \delta_{K_n}, K_n\rangle + n).\alpha(\varepsilon)q\varepsilon). \tag{113}$$

Below we will study this inductively determined sequence of differential equations. We shall obtain polynomial solutions for these equations. The way they are obtained differs according to the fact that $\mathcal{S}(K_n)$ is zero or nonzero.

Definition 5. If $\mathcal{S}(K_n) = 0$ we will say that τ_{K_n} is obtained from $\tau_{K_{n-1}}$ by an arrow $\tau_{K_{n-1}} \to \tau_{K_n}$ *of type A*. If $\mathcal{S}(K_n) \neq 0$ we will say that such an arrow $\tau_{K_{n-1}} \to \tau_{K_n}$ is *of type B*.

8.3.1 An arrow of type A

Let $D \geq 1$ be a given integer and let

$$\tau = \sum_{j=0}^{D-1} a_j \sigma^j$$

where $a_j = a_j(\varepsilon)$ is a function of ε. Let us for simplicity pass to the new small parameter $\eta = \alpha(\varepsilon)q\varepsilon$. We look for $\hat{\tau}$ such that

$$\frac{d\hat{\tau}}{d\sigma}.(1+\eta\sigma) = \tau + \hat{\tau}.D\eta, \tag{114}$$

see equation (113) with $\mathcal{S}(K_n) = 0$ and $D = -\langle \delta_{K_n}, K_n \rangle + n$. We look for $\hat{\tau}$ of the form

$$\hat{\tau} = \sum_{j=1}^{D} \hat{a}_j \sigma^j.$$

Inserting this in both sides of equation (114) gives, after some easy calculation:

$$j!\hat{a}_j = \frac{1}{(D-j)!} \sum_{i=0}^{j-1} (D-i-1)! \eta^{j-1-i} i! a_i \tag{115}$$

for all $1 \leq j \leq D$. We can consider this as a linear map $\mathbb{C}^D \to \mathbb{C}^D$ mapping the $i!a_i$ to the $j!\hat{a}_j$, as defined by (115), that is: $(0!a_0, \ldots, i!a_i, \ldots, (D-1)!a_{D-1}) \mapsto (1!\hat{a}_1, \ldots, j!\hat{a}_j, \ldots, D!\hat{a}_D)$. It has a $D \times D$ matrix $\hat{C} = (\hat{C}_{jk})_{1 \leq j,k \leq D}$ with

$$\hat{C}_{jk} = \begin{cases} \frac{(D-k)!}{(D-j)!} \eta^{j-k} \text{ if } j \geq k \\ 0 \text{ if } j < k. \end{cases}$$

8.3.2 An arrow of type B

Let $\tau = \sum_{j=0}^{d} a_j \sigma^j$ where $a_j = a_j(\varepsilon)$ is a function of ε. We look for $\tilde{\tau}$ such that

$$\frac{d\tilde{\tau}}{d\sigma}.(1+\alpha(\varepsilon)q\varepsilon\sigma) = \tau + \tilde{\tau}.(S_\varepsilon + D\alpha(\varepsilon)q\varepsilon) \tag{116}$$

where we assume that $S_\varepsilon \neq 0$. We think of $S_\varepsilon = \mathcal{S}(K_n)\alpha(\varepsilon)(1+r_0\varepsilon)$ and $D = -\langle \delta_{K_n}, K_n \rangle + n$, see equation (113). We look for $\tilde{\tau}$ of the form

$$\tilde{\tau} = \sum_{j=0}^{d} \tilde{a}_j \sigma^j.$$

Inserting this in both sides of equation (116) readily yields the following 'downward' inductive scheme:

$$\begin{aligned} -\tilde{a}_d &= \frac{1}{S_\varepsilon + (D-d)\alpha(\varepsilon)q\varepsilon}.a_d \\ -\tilde{a}_{d-1} &= \frac{1}{S_\varepsilon + (D-d+1)\alpha(\varepsilon)q\varepsilon}.(d(-\tilde{a}_d) + a_{d-1}) \\ &\vdots \\ -\tilde{a}_j &= \frac{1}{S_\varepsilon + (D-j)\alpha(\varepsilon)q\varepsilon}.((j+1)(-\tilde{a}_{j+1}) + a_j) \\ &\vdots \\ -\tilde{a}_0 &= \frac{1}{S_\varepsilon + D\alpha(\varepsilon)q\varepsilon}.(-\tilde{a}_1 + a_0), \end{aligned} \tag{117}$$

that is: we solve this successively from top to bottom, determining all the $\tilde{a}_j$.

Remark 11. If one looks for a 'simpler' majorant for this arrow B, and if one assumes that for small ε all denominators in (117) are estimated from below, that is: suppose there exists an $S > 0$ such that

$$|S_\varepsilon + (D-j)\alpha(\varepsilon)q\varepsilon| \geq S > 0$$

for all $j = 0, \ldots, d$, then a calculation shows that

$$-\tilde{a}_j = \sum_{i=j}^{d} \frac{1}{S^{i-j+1}} \frac{i!}{j!} a_i$$

provides a majorant. This could perhaps be helpful in answering the pending question that we pose below in paragraph 8.3.4.

8.3.3 Conclusion of the proof of Theorem 9

We recall Notation 3, especially $D(K_n) = -\langle \delta_{K_n}, K_n \rangle + n$; observe that $D(K_n) \leq |K_n| + n$.

Lemma 9. *We have:*
(i) ω_{K_n} *is a polynomial of degree at most* $D(K_n)$*,*
(ii) $v_{K_n} = x^\alpha y^\beta$ *with* $\alpha + \beta \geq |K_n|$*.*

Proof. (i) We have for all $n \geq 1$

$$\begin{aligned} D(K_{n+1}) &= -\langle \delta_{K_{n+1}}, K_{n+1} \rangle + n + 1 \\ &= D(K_n) - \delta_{k_{n+1}} k_{n+1} + 1 \\ &\geq D(K_n) + 1 \end{aligned}$$

and hence the sequence $(D(K_n))_{n \geq 1}$ is strictly increasing. The statement in (i) is clearly true for $n = 1$. Assume, by induction, that (i) holds for $n \geq 1$. Recall τ_{K_n} from (112). We obtain $\tau_{K_{n+1}}$ either by an arrow of type A or B. For an arrow of type A we have to take $D = D(K_{n+1})$ and $\tau = \tau_{K_n}$. The degree of τ is indeed at most $D(K_n) \leq D(K_{n+1}) - 1 = D - 1$, as it should be, and the degree of $\tau_{K_{n+1}} = \tilde{\tau}$ is then at most $D(K_{n+1})$. For an arrow of type B, the degree does not change. This finishes the induction for (i).

(ii) Recall the definition (36) of v_k. First of all, since $(r_0, s_0) \neq (0,0) \neq (r_1, s_1)$, one has $v_k = x^\alpha y^\beta$ with $\alpha + \beta \geq |k|$. Statement (ii) now immediately follows by induction from (38). ∎

We claim that a term $\omega_{K_n}(s)v_{K_n}(F_{K_n}, G_{K_n})(u,\varepsilon)$ can be written in the form (41). Indeed, by Lemma 9, such a term is of order $\omega_{K_n}(s)v_{K_n}.O(u^n) = \omega_{K_n}(s)x^\alpha y^{|K_n|-\alpha}u^n O(1)$ for some integer $\alpha \geq 0$. For each $0 \leq j \leq D(K_n)$ we can decompose a monomial $s^j x^\alpha y^{|K_n|-\alpha}u^n$, with $0 \leq j \leq D(K_n)$, as $s^j x^\alpha y^{|K_n|-\alpha}u^n = (sx)^{\tilde{\alpha}}.(sy)^{\tilde{\beta}}(su)^{\tilde{n}}x^{\alpha-\tilde{\alpha}}y^{|K_n|-\alpha-\tilde{\beta}}u^{n-\tilde{n}}$, where all the occurring exponents are integers ≥ 0. This is possible since $j \leq D(K_n) \leq |K_n| + n$. (Remark that such a decomposition is not unique; in order to fix the ideas we give an example: $s^8x^2y^4u^3 = (sx)^2(sy)^4(su)^2u = (sx)(sy)^4(su)^3x$.) ∎

8.3.4 Digression and question about Theorem 9

Remark 12. (i) The domain for ε depends on K_n: see (117).
(ii) The form in (41) is not unique.

We end this paper with a question:

Remark 13. From the theorem it follows that $(\tilde{x}, \tilde{y})$ in (40) is a formal power series in (sx, sy, x, y), and it would be interesting to investigate its asymptotics, in a similar manner as in sections 3.3 and 6.4.

The polynomials ω_{K_n} are obtained from equation (110) in a recursive way using arrows of type A or B. They are 'universal' in the sense that they only depend on p and q, not on the family of vector fields we start from. Furthermore, the functions (F_{K_n}, G_{K_n}) are generated by an explicit recursion, as can be seen in the formulas (106) - (109).

Despite the explicit nature of this procedure, we did not succeed in computing a satisfactory majorant (for instance of Gevrey type) in a comparable way as in Theorem 7, nor as in [5, section 5.4]. Let us indicate what are the difficulties that we encounter here. In section 6.2 we only had to consider functions of the compensator for the special case $K_n = (0, \ldots, 0)$, that is: $\omega_{K_n}(s_\varepsilon) = s_\varepsilon^n/n!$. For a general K_n, the expressions obtained from equation (110) are less understood. Moreover, for $\varepsilon \neq 0$ we cannot invoke the method of [5, page 1153, Lemma 1], stating that an arrow of type B followed by an arrow of type A is majorated by reversing the two, which is no longer true here, and which could have made the estimates manageable. It would be interesting to get more insight in this issue. One aspect of this is a better understanding of the 'universal' (multi-)series of the ω_{K_n}, compare to [5, inequality (71)].

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