Dynamics of multidimensional Cesàro operators

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Abstract

We study the dynamics of the multi-dimensional Cesàro integral operator on $L^p(I^n)$, for I the unit interval, $1 < p < \infty$, and $n \geq 2$, that is defined as

$$\mathcal{C}(f)(x_1, \ldots, x_n) = \frac{1}{x_1 x_2 \cdots x_n} \int_0^{x_1} \ldots \int_0^{x_n} f(u_1, \ldots, u_n) du_1 \ldots du_n \quad \text{for } f \in L^p(I^n).$$

This operator is already known to be bounded. As a consequence of the Eigenvalue Criterion, we show that it is hypercyclic as well. Moreover, we also prove that it is Devaney chaotic and frequently hypercyclic.

1 Introduction

A *Cesàro integral operator* is a particular case of a Volterra type operator on function spaces that is defined as the average:

$$(\mathcal{C}f)(x) := \frac{1}{x} \int_0^x f(s) ds \tag{1.1}$$

In this line, the discrete version of this operator considered on sequence spaces is defined as

$$(x_1, x_2, \ldots) \to \left(x_1, \frac{x_1 + x_2}{2}, \frac{x_1 + x_2 + x_3}{3}, \ldots, \frac{x_1 + \ldots + x_n}{n}, \ldots \right). \tag{1.2}$$

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Volterra type operators (and, in particular, Cesàro operators) have been widely studied in linear dynamics. We recall that an operator $T \in L(X)$ is *hypercyclic* either if or as long as there is some $x \in X$ such that $\{T^n x : n \in \mathbb{N}\}$ is dense in X, and T is *supercyclic* if $\{\lambda T^n x : \lambda \in \mathbb{C}, n \in \mathbb{N}\}$ is dense in X, too. Related to this, *Cesàro hypercyclicity* consists on the density of the Cesàro means of a given operator $T \in L(X)$ for some element in the space X, i.e. there exists some $x \in X$ such that $\{n^{-1}(\sum_{k=0}^{n} T^k x) : n \in \mathbb{N}\}$ is dense in X. We recall that $T \in L(X)$ is said to be *Devaney chaotic* if it is hypercyclic and it has a dense set of periodic points, and $T \in L(X)$ is said to be *frequently hypercyclic* if there exists some $x \in X$ such that the set $N(x, U) := \{n \in \mathbb{N} : T^n x \in U\}$ has positive lower density, c.f. [13]. Further information concerning linear dynamics can be found in [1, 21, 27].

The study of the dynamics of the Cesàro means was considered by León-Saavedra in [36]. Its study is motivated by some questions coming from ergodic theory [25, 38, 40, 42]. Cesàro hypercyclicity is equivalent to the density of $\{n^{-1} T^n x : n \in \mathbb{N}\}$ for some $x \in X$, which yields that Cesàro hypercyclicity is a special kind of supercyclicity. The classes of Cesáro-hypercyclic operators and of hypercyclic operators have nonempty intersection but neither is contained in the other [36], see also [22, 23]. Cesàro hypercyclicity for weighted shifts was considered in [36]. The existence of common Cesàro hypercyclic vectors has been analyzed in [24].

Let $I = [0, 1]$. The dynamics of the Cesàro integral operator has been considered on the spaces $L^p(I)$, with $1 < p < \infty$, and on $C(I)$, the space of continuous functions on I endowed with the supremum norm. More precisely, it is known to be hypercyclic on $L^p(I)$, with $1 < p < \infty$, as a consequence of the Full Müntz-Szász theorem [26] and of the Eigenvalue Criterion, see below. Moreover, it is Devaney chaotic, it contains hypercyclic subspaces, but it is not hyponormal. In contrast, on $C(I)$ it is not even supercyclic [37]. This last statement can be compared with a previous result of González & León-Saavedra who showed that, despite being cyclic, the Cesáro integral operator is not supercyclic, nor weakly supercyclic on $L^2(\mathbb{R}_+)$ [30].

Power bounded and mean ergodic properties of the Cesàro integral operator and their connection with hypercyclicity have been recently considered by Albanese et al in [3, 5], as well as the hypercyclicity of discrete Cesàro operator [4, 6, 8, 7, 8].

We study the dynamics of the multi-dimensional Cesáro integral operator on $L^p(I^n)$, for $1 < p < \infty$ and $n \geq 2$, that is defined as

$$\mathcal{C}(f)(x_1, \ldots, x_n) = \frac{1}{x_1 x_2 \cdots x_n} \int_0^{x_1} \cdots \int_0^{x_n} f(u_1, \ldots, u_n) du_1 \ldots du_n, \; f \in L^p(I^n). \tag{1.3}$$

This operator is already known to be bounded, cf. [20]. As a consequence of the Eigenvalue Criterion, we will show that it is hypercyclic. Besides, we also prove that it is Devaney chaotic and even frequently hypercyclic.

2 Main results

Let $\Omega := \{z \in \mathbb{C} : \Re(z) > -1/p\}$ for $1 < p < \infty$. We first observe that if $\mathcal{C}$ acts on $L^p(I^n)$, $n > 2$, then

$$\mathcal{C}(x_1^{\alpha_1} \cdots x_n^{\alpha_n}) = \frac{1}{(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_n+1)} x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \tag{2.1}$$

for every $(\alpha_1, \ldots, \alpha_n) \in \Omega^n$.

The spectrum of this operator was known thanks to Leibowitz [34, 35], see also [19]. The abundance of these eigenvectors will permit us to determine the dynamics of the Cesàro integral operator. The following result is inspired in the Hypercyclicity Criterion. Its formulation is expressed in terms of the abundance of eigenvectors of $\mathcal{C}$ [11, 33].

Theorem 2.1. *(Eigenvalue Criterion) Let $T \in L(X)$ be a bounded operator on a complex Banach space X. If the sets*

$$\operatorname{span} \bigcup_{|\lambda|<1} \ker(T - \lambda I) \text{ and } \operatorname{span} \bigcup_{|\lambda|>1} \ker(T - \lambda I) \tag{2.2}$$

are dense in X, then T is hypercyclic.

This result was later improved by Grivaux, just restricting to the part of the spectrum contained in the unit circle S^1 [32], see also [14].

Theorem 2.2. *(Eigenvalue Criterion for Frequent Hypercyclicity) Let $T \in L(X)$ be a bounded operator on a complex Banach space X. If the set*

$$\operatorname{span} \bigcup_{\lambda \in S^1 \setminus D} \ker(T - \lambda I), \tag{2.3}$$

is dense in X for every countable subset $D \subset S^1$, then T is frequently hypercyclic.

In order to apply the previous results to $\mathcal{C}$, we will take into account the following facts:

Theorem 2.3. *[41, p. 32]. The zero set of an analytic function in $n \geq 2$ variables is never discrete if it is non-empty. In particular, it has a limit point.*

The lack of analyticity can permit the existence of an abundance of surprising everywhere surjective functions where the zero set can have nearly any admissible description [9].

Theorem 2.4. *If $g \in L^p(I)$ and $\int_0^1 g(x)x^\lambda dx = 0$ for all λ in a set with an accumulation point, then $g = 0$ identically.*

Let $\varphi : \Omega^n \to \mathbb{C}$ be the analytic map

$$\varphi(z_1, \ldots, z_n) = \frac{1}{(z_1+1)(z_2+1)\cdots(z_n+1)}. \tag{2.4}$$

The next lemma is based on the nice surjective properties of this function on several complex variables and is a generalization of Theorem 2.4.

Lemma 2.5. *Let $n \geq 1$. If $\Gamma \subset \varphi(\Omega^n)$ with an accumulation point, then*

$$\operatorname{span}\{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} : (\alpha_1, \ldots, \alpha_n) \in \varphi^{-1}(\Gamma)\} \tag{2.5}$$

is dense in $L^p(I^n)$, where $1 < p < \infty$.

Proof. We can assume $n \geq 2$, since the result is essentially known for $n = 1$. We will prove that if $f \in L^p(I^n)$ where $1/p + 1/q = 1$ and

$$\int_0^1 \int_0^1 \cdots \int_0^1 f(x_1, x_2, \ldots, x_n) x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} dx_n \ldots dx_1 = 0 \tag{2.6}$$

for all $(\alpha_1, \ldots, \alpha_n) \in \varphi^{-1}(\Gamma)$, then $f(x_1, \ldots, x_n) = 0$.

Fix $\lambda \in \Gamma$. We first point out the symmetry of the sets $\varphi^{-1}(\lambda)$. Take $(\alpha_1, \ldots, \alpha_n) \in \Omega^n$ enjoying

$$\frac{1}{(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_n + 1)} = \lambda. \tag{2.7}$$

By symmetry, if we take any permutation over a set of n elements, $\sigma \in \Sigma_n$, we have $(\alpha_1, \ldots, \alpha_n) \in \varphi^{-1}(\lambda)$ if and only if $(\alpha_{\sigma^{-1}(1)}, \ldots, \alpha_{\sigma^{-1}(n)}) \in \varphi^{-1}(\lambda)$. Combining this with Theorem 2.3, we can certainly find a set $\Theta_1 \subset \Omega$ having a limit point, and numbers $\alpha_{1j} \in \Omega$, $2 \leq j \leq n$, such that

$$\frac{1}{(z_1 + 1)(\alpha_{12} + 1) \ldots (\alpha_{1n} + 1)} = \lambda \text{ for all } z_1 \in \Theta_1. \tag{2.8}$$

This can be done by considering the projections onto each coordinate, and taking into account that if there is a limit point, then at least, this can be seen through one of the projections.

Now fix $z_1 \in \Theta_1$. From (2.8), we have an analytic function in $n - 1$ variables and hence, using the same argument as above, we find a set $\Theta_2 \subseteq \Omega$ with a limit point such that

$$\frac{1}{(z_1 + 1)(z_2 + 1)(\alpha_{23} + 1) \ldots (\alpha_{2n} + 1)} = \lambda \text{ for all } z_2 \in \Theta_2, \tag{2.9}$$

for some $\alpha_{2j} \in \Omega$, $3 \leq j \leq n$. Note that Θ_2 depends on z_1. Fixing $z_2 \in \Theta_2$, we repeat the process in order to get $\Theta_3, \ldots, \Theta_{n-1} \subseteq \Omega$ with accumulation points. We cannot go further with the previous argument to find Θ_n, since the zero set of a certain nonzero one variable analytic map is always discrete.

For finding such a set $\Theta_n \subset \Omega$ with a limit point, we will use the fact that our given set Γ is having a limit point in $\varphi(\Omega^n)$. Our recursively chosen element $(z_1, \ldots, z_{n-1})$ verifies

$$\frac{1}{(z_1 + 1)(z_2 + 1) \ldots (z_{n-1} + 1)(\alpha_{nn} + 1)} = \lambda, \tag{2.10}$$

for some $\alpha_{nn} \in \Omega$. Consider the open set

$$\frac{1}{(z_1 + 1)(z_2 + 1) \ldots (z_{n-1} + 1)(\Omega + 1)},$$

which contains λ as an accumulation point. Hence there exists a subset $\Gamma' \subseteq \Gamma$, with also λ as a limit point, such that

$$\Gamma' \subset \frac{1}{(z_1+1)(z_2+1)\dots(z_{n-1}+1)(\Omega+1)}.$$

As λ is non-zero, we find Θ_n with limit points such that, for each $z_n \in \Theta_n$, we have

$$\frac{1}{(z_1+1)(z_2+1)\dots(z_{n-1}+1)(z_n+1)} \in \Gamma' \subseteq \Gamma. \tag{2.11}$$

Indeed, one can define Θ_n as

$$\Theta_n = \frac{1}{(z_1+1)\dots(z_{n-1}+1)\Gamma'} - 1, \tag{2.12}$$

and it has limit points as $\lambda \neq 0$.

Consider the recursively chosen points $z_1, \dots, z_n$ with z_n varying in Θ_n and rewrite equation (2.6) as

$$\int_0^1 \left(\int_0^1 \dots \int_0^1 f(x_1, x_2, \dots, x_n) x_1^{z_1} x_2^{z_2} \dots x_{n-1}^{z_{n-1}} dx_1 \dots dx_{n-1} \right) x_n^{z_n} dx_n = 0. \tag{2.13}$$

By Theorem 2.4 , the integral inside brackets vanishes for all $x_n \in I$. Rewrite this $n-1$ multiple integral in the same fashion as before and vary z_{n-1}. Continuing in this way, we get

$$\int_0^{x_1} f(x_1, x_2, \dots, x_n) x_1^{z_1} dx_1 = 0 \text{ for all } x_2, x_3, \dots, x_n. \tag{2.14}$$

Again, as z_1 is an arbitrary point in Θ_1, which has limit points in Ω, we conclude $f(x_1, \dots, x_n) = 0$ in all coordinates. ∎

From the above lemma, our main result follows:

Theorem 2.6. *The operator $\mathcal{C}$ is frequently hypercyclic and Devaney chaotic on $L^p(I^n)$, for $1 < p < \infty$.*

Proof. To prove the frequent hypercyclicity, we show that the span of all eigenvectors corresponding to the eigenvalues in a set $S^1 \setminus D$ is dense in $L^p(I^n)$ for all countable subsets $D \subseteq S^1$. Recall that the eigenvalues of $\mathcal{C}$ are of the form $\varphi(z_1, \dots, z_n)$, with $(z_1, \dots, z_n) \in \Omega^n$. There is some arc $S \subset S^1$ with $S \subset \sigma_p(\mathcal{C}) \cap \varphi(\Omega^n)$. The set $S \setminus D$ is uncountable, and then it contains some of its limit points. Thus, by Lemma 2.5 the operator $\mathcal{C}$ satisfies the Eigenvalue Criterion.

To show the Devaney chaos, we only have to check that the periodic points of $\mathcal{C}$ form a dense subspace of $L^p(I^n)$. Indeed, choose some rational sequence $\{\theta_k\}_k$ with distinct terms in $[0,1]$ so that $\{e^{2\pi i \theta_k}\}_k$ converges to 1. Setting $\Gamma = \{e^{2\pi i \theta_k} : k \geq 1\}$, we see that $\mathcal{C}$ is Devaney chaotic.

∎

The quest for infinite-dimensional closed subspaces all of whose non-zero vectors are hypercyclic was raised in [17]. Since then, it has been one of the main topics of lineability theory [1, 2, 18]. In order to prove the existence of subspaces of frequently hypercyclic vectors for $\mathcal{C}$, we will make use of the following result, which is inspired in the criterion for subspaces of hypercyclic vectors stated by Montes [39], see also [31], and in the Frequent Hypercyclicity Criterion [12, 13, 15].

Theorem 2.7. *[16] Suppose that T satisfies the Eigenvalue Criterion on a complex Banach space X. Let $E \subset X$ be a closed infinite dimensional subspace E such that $\lim_{k\to\infty} T^k x = 0$ for all $x \in E$. Then T has a frequently hypercyclic subspace.*

In particular this is so if $\ker(T - \mu I)$ is infinite dimensional for some $|\mu| < 1$.

Theorem 2.8. *The operator $\mathcal{C}$ admits frequently hypercyclic subspaces on $L^p(I^n)$, with $1 < p < \infty$.*

Proof. Note that $\mathcal{C}$ satisfies the Eigenvalue Criterion in $L^p(I^n)$. We first consider the case $n \geq 2$ and prove that $\ker(\mathcal{C} - \mu I)$ is infinite dimensional for some $|\mu| < 1$. Indeed, as the function φ is analytic on Ω^n, it is an open map. As $1 \in \varphi(\Omega^n)$, there exists $(a_1, \dots, a_n)$ and $|\mu| < 1$ such that $\varphi(\alpha_1, \dots, \alpha_n) = \mu$.

$$\frac{1}{(\alpha_1 + 1) \cdots (\alpha_n + 1)} = \mu. \tag{2.15}$$

Again, by Theorem 2.3, the solution set of the above equation (i.e. the zero set of the analytic function $\varphi - \mu$) is never discrete and so, it is an infinite set. Now, if $(\alpha_1, \dots, \alpha_n) \neq (\beta_1, \dots, \beta_n)$ satisfying (2.15), then the eigenfunctions $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $x_1^{\beta_1} \cdots x_n^{\beta_n}$ are linearly independent. Hence $\ker(\mathcal{C} - \mu I)$ is infinite dimensional.

Next, we consider the case $n = 1$. Let $(a_k)_k$ be an infinite sequence such that

$$\mathcal{M}_a := \text{span}\left\{x^{a_k} : k \geq 1\right\} \tag{2.16}$$

is a proper subspace of $L^p(I)$, where the closure is taken in the $|| \cdot ||_p$ norm. Such a space is called a *Müntz space*; see [29] for details. By [29, Cor. 6.2.4], if $(a_k)_k$ is an strictly increasing sequence of positive numbers with $\sum_{k=1}^{\infty} \frac{1}{a_k} < \infty$ and $\inf_k(a_{k+1} - a_k) > 0$, then every element $f \in \mathcal{M}_\alpha$ has an analytic expansion

$$f(x) = \sum_{k=1}^{\infty} c_k x^{a_k} \tag{2.17}$$

in $[0, 1)$ and in particular, f is continuous on $[0, 1)$. (For example, one can choose $a_k := k^2$). Observe that $f(0) = 0$ for all $f \in \mathcal{M}_a$. Now consider the subspace

$$Y_a = \left\{\chi_{[0,1/2]} f : \, f \in \mathcal{M}_a\right\}, \tag{2.18}$$

which is an infinite dimensional, closed subspace in $L_p[0, 1]$. Moreover, we have $\mathcal{C}(Y_a) \subset C[0, 1]$. By a result of Galaz-Fontes and & Solís [28], we have that $\mathcal{C}^k(\mathcal{C}f)$ converges to the constant function $\mathcal{C}f(0)$ in supremum norm for all $f \in C[0, 1]$. But, for all $f \in Y_a$, we have that $\mathcal{C}f$ is continuous and vanishes at 0. Thus, the sequence $(\mathcal{C}^k(f))_k$ converges to 0 in the p-norm for all $f \in Y_a$. The conclusion is obtained by an application of Theorem 2.7. ∎

We conclude with some remarks:

Remark 2.9. The Cesàro operator acting on $L^p(I^n)$ is also Devaney chaotic and frequently hypercyclic for all $0 < p \leq 1$, which follows as the embedding of $L^2(I^n)$ into $L^p(I^n)$ is continuous.

Remark 2.10. The Cesàro operator satisfies the Eigenvalue Criterion in $L_p(I^n)$, $1 < p < \infty$. Hence it is topologically mixing on all $L^p(I^n)$. Moreover, for any strictly increasing sequence of natural numbers $(m_k)_k$, the family $\{\mathcal{C}^{m_k} : k \geq 1\}$ is hypercyclic on these spaces.

Remark 2.11. In the same way as in [3, Th. 4.2], by using [10, Th.2.1], the results obtained in $L^p(I^n)$ can be extended to the Fréchet spaces $L^p_{loc}(\mathbb{R}^+)$ with $1 < p < \infty$, consisting of all complex valued measurable functions on positive real semi-axis which are p-th power integrable on each interval $[0, j]$, for all $j \in \mathbb{N}$. Extensions to multidimensional Fréchet spaces could also be considered.

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