Divergent series of Taylor coefficients on almost all slices

Piotr Kot Marek Karaś

Abstract

We show that there exists a holomorphic function, continuous to the boundary in a bounded, balanced, strictly pseudoconvex domain Ω with C^2 boundary such that almost every slice function has a series of Taylor coefficients divergent with every power $p \in (0, 2)$.

1 Introduction

1.1 Historical background.

In [10, 7.2] Rudin gives some examples of boundary behavior of holomorphic functions in the unit balls of dimensions 2 and 3. Ryll and Wojtaszczyk observed [8, Theorem 1.2 + Remark 1.10] that similar examples can be constructed in arbitrary dimension. The crucial tool used in reminded constructions is [8, Theorem 1.2]: there exist polynomials $\{p_n\}$ homogeneous of degree n on the unit ball B^d such that

$$\|p_n\|_2 = 1 \text{ and } \|p_n\|_\infty \leq \frac{2^d}{\sqrt{\pi}}. \tag{1.1}$$

This tool can be used to convert some one dimensional examples into multidimensional cases. An interesting example of such an application is presented in paper [7].

It is known that there exists a holomorphic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in the disk-algebra and such that $\sum_{n=0}^{\infty} |a_n|^p = \infty$ for all $p < 2$. Wojtaszczyk generalized this fact.

Received by the editors in August 2017 - In revised form in February 2018.
Communicated by H. De Bie.
2010 *Mathematics Subject Classification* : Primary 32A40; Secondary 32A05, 32E35.
Key words and phrases : Inner Function, Taylor coefficients,

1.2 Motivations.

Our inspiration is paper [7], where it was proven that there exists a function $f \in A(\mathbb{B}^d)$ such that almost every slice function of f has a series of Taylor coefficients divergent with every power $p < 2$.

We are going to strengthen the Wojtaszczyk's result [7] by showing that:

1. the unit ball $\mathbb{B}^d$ can be replaced by Ω bounded, balanced, strictly pseudoconvex domain with C^2 boundary,
2. it is possible to construct f in the elementary way.

We use a fact [2, Theorem 3.2] about commonly bounded holomorphic functions which are big in each boundary point. Let us note that [2, Theorem 3.2] can be used in the construction of an inner function (see [2]).

Wojtaszczyk uses (1.1) in [7, Proposition] to describe surjectivity of the operator

$$T : A(\mathbb{B}^d) \ni f \to (\langle f, p_{2^n} \rangle)_{n=0}^{\infty} \in l_2$$

by duality theorem. In one variable, the constructive proof of such surjectivity can be found in [1]. As Wojtaszczyk mentioned it would be interesting to have such a constructive proof also in the case of the unit ball $\mathbb{B}^d$. We don't know if similar result can be obtained for other domains than $\mathbb{B}^d$. Wojtaszczyk uses also "scrambling lemma", which needs unitary mappings $A(\mathbb{B}^d)$. A lack either of the mentioned surjectivity or "scrambling lemma" does not enable to generalize the Wojtaszczyk's proof.

1.3 Notations.

Let Ω be a bounded, balanced, strictly pseudoconvex domain with the boundary of class C^2. Now we denote σ as a standard circular invariant measure on $\partial\Omega$ with $\sigma(\partial\Omega) = 1$.

Given $f \in A(\Omega)$ we study the slice function $\mathbb{B}^1 \ni \lambda \to f(\lambda z)$ and the middle value $\|f\|_z := \sqrt{\int_0^1 |f(e^{2\pi i t} z)|^2 dt}$ of holomorphic function f on a circle given by the point $z \in \partial\Omega$.

We need the following fact:

Theorem 1. *(see [2, Theorem 3.2], [5, Lemma 2.1]). Let $m \in \mathbb{N}$. There exists a natural number $N_0 = N_0(\partial\Omega)$ such that, if $\varepsilon \in (0,1)$, h is a continuous, strictly positive function on $\partial\Omega$, then there exist polynomials $f_1, ..., f_{N_0} \in A(\Omega)$ such that:*

1. *each nonzero term in the expansion of f_j (for all j) has a degree greater than m,*
2. *$|f_j| < h$ on $\partial\Omega$,*
3. *$\frac{1}{2}h < \max_{j=1,...,N_0} |f_j|$ on $\partial\Omega$.*

The theorem above is proved in a more general situation e.g. for a domain with Holomorphic Support Function but we consider here only a simplified version for a strictly pseudoconvex case.

1.4 Main result.

We obtain the following fact:

Theorem. *Assume that Ω is a bounded, balanced, strictly pseudoconvex domain with the boundary of class C^2. There exists a holomorphic function $f \in A(\Omega)$ such that almost every slice function has a series of Taylor coefficients divergent with every power $p \in (0, 2)$.*

To obtain Taylor series of a function f it is sufficient to find a homogeneous expansion:

$$f(z) = \sum_{n=0}^{\infty} p_n(z)$$

where p_n is a homogeneous polynomial of a degree n. Now we have Taylor coefficients expansion for a slice function:

$$\lambda \to f(\lambda z) = \sum_{n=0}^{\infty} p_n(z)\lambda^n,$$

so we construct a holomorphic function $f \in A(\Omega)$ with:

$$\sum_{n=0}^{\infty} |p_n(z)|^s = \infty$$

for $s \in (0, 2)$ and σ-almost all $z \in \partial\Omega$. Note that if f is continuous to the boundary, then (for all $z \in \partial\Omega$):

$$\sum_{n=0}^{\infty} |p_n(z)|^2 < \infty.$$

2 Holomorphic functions with divergent taylor series

Lemma 2. *There exists a constant $\gamma > 0$ such that for $\kappa \in \mathbb{N}$, $\tilde{\varepsilon} > 0$, and a positive, continuous function h on $\partial\Omega$ we can choose a polynomial p and a compact subset K of $\partial\Omega$ such that:*

- *each nonzero monomial in p has a degree greater than κ,*
- $|p| < h$ *on* $\partial\Omega$,
- $\|p\|_z \geq \gamma \|h\|_z$ *for* $z \in K$,
- $\sigma(K) > 1 - \tilde{\varepsilon}$.

Proof. Let $\gamma > 0$ be such that $\gamma = \frac{1}{2\sqrt{N_0}}(1-\gamma)$ where $N_0 \in \mathbb{N}$ is the constant from Theorem 1. We construct a sequence of polynomials p_n such that we have the following conditions fulfilled:

1. each nonzero term in the expansion of p_n has a degree greater than κ and less than the degree of each term in the expansion of p_{n+1},
2. $|\sum_{k=1}^{n} p_k| < h$ on $\partial\Omega$,
3. if $n > 1$ then the compact set $K_n := \left\{z \in \partial\Omega : \|\sum_{k=1}^{n} p_k\|_z \geq \gamma \|h\|_z\right\}$ has the following properties:
   (a) $K_n \subset K_{n+1}$,
   (b) $\sigma(K_{n+1} \setminus K_n) \geq \frac{1}{2N_0}\sigma(\partial\Omega \setminus K_n)$.

Let $p_1 := 0$. Then $K_1 = \emptyset$ and the conditions (1)-(2) are fulfilled. Now suppose that we have chosen $p_1, ..., p_n$ according to (1)-(3). Due to the Theorem 1 there exist polynomials $g_1, ..., g_{N_0}$ such that:

- each monomial in g_j has a degree greater than monomials' degrees in $p_1, ..., p_n$,
- $|g_j| < h - |\sum_{k=1}^{n} p_k|$ on $\partial\Omega$,
- $\frac{1}{2}(h - |\sum_{k=1}^{n} p_k|) < \max_{j=1,...,N_0} |g_j|$ on $\partial\Omega$.

If $z \in \partial\Omega$ then

$$\sum_{j=1}^{N_0} \|g_j\|_z^2 = \sum_{j=1}^{N_0} \int_0^1 \left|g_j\left(e^{2\pi it}z\right)\right|^2 dt \geq \int_0^1 \max_{j=1,...,N_0} \left|g_j\left(e^{2\pi it}z\right)\right|^2 dt$$
$$\geq \int_0^1 \frac{1}{4}\left|\left(h - \left|\sum_{k=1}^{n} p_k\right|\right)\left(e^{2\pi it}z\right)\right|^2 dt = \frac{1}{4}\left\|h - \left|\sum_{k=1}^{n} p_k\right|\right\|_z^2.$$

In particular there exists $j_z \in \{1, ..., N_0\}$ such that

$$\|g_{j_z}\|_z^2 \geq \frac{1}{4N_0}\left\|h - \left|\sum_{k=1}^{n} p_k\right|\right\|_z^2$$

Now we can define

$$V_j := \left\{z \in \partial\Omega \setminus K_n : \|g_j\|_z^2 \geq \frac{1}{4N_0}\left\|h - \left|\sum_{k=1}^{n} p_k\right|\right\|_z^2\right\}.$$

and observe that $\partial\Omega \setminus K_n = \bigcup_{j=1}^{N_0} V_j$. In particular there exists $j \in \{1, ..., N_0\}$ such that $\sigma(V_j) \geq \frac{1}{N_0}\sigma(\partial\Omega \setminus K_n)$. We can choose a compact set $T \subset V_j$ such that $\sigma(T) \geq \frac{1}{2N_0}\sigma(\partial\Omega \setminus K_n)$.

We define $p_{n+1} = g_j$ and observe that p_{n+1} fulfills the properties (1)-(2).

Let us consider $K_{n+1} = \left\{z \in \partial\Omega : \left\|\sum_{k=1}^{n+1} p_k\right\|_z \geq \gamma \|h\|_z\right\}$. Since $p_1, ..., p_n, p_{n+1}$ are orthogonal in an L^2 space on slices i.e. $\left\|\sum_{k=1}^{n+1} p_k\right\|_z^2 = \sum_{k=1}^{n+1} \|p_k\|_z^2$ for all

$z \in \partial\Omega$, we can easily observe that $\left\|\sum_{k=1}^{n+1} p_k\right\|_z \geq \|\sum_{k=1}^{n} p_k\|_z$ for all $z \in \partial\Omega$, which implies that $K_n \subset K_{n+1}$.

Let $z \in T$. Since $T \subset \partial\Omega \setminus K_n$ we have $\|\sum_{k=1}^{n} p_k\|_z < \gamma \|h\|_z$ which implies

$$\|p_{n+1}\|_z = \|g_j\|_z \geq \sqrt{\frac{1}{4N_0}} \left\| h - \left| \sum_{k=1}^{n} p_k \right| \right\|_z \geq \frac{1}{2\sqrt{N_0}} \left(\|h\|_z - \left\| \sum_{k=1}^{n} p_k \right\|_z \right)$$
$$\geq \frac{1}{2\sqrt{N_0}} \left(\|h\|_z - \gamma \|h\|_z\right) \geq \frac{1}{2\sqrt{N_0}} (1-\gamma) \|h\|_z = \gamma \|h\|_z ,$$

but $\left\|\sum_{k=1}^{n+1} p_k\right\|_z \geq \|p_{n+1}\|_z$, so $T \subset K_{n+1}$. In particular

$$\sigma(K_{n+1} \setminus K_n) \geq \sigma(T) \geq \frac{1}{2N_0} \sigma\left(\partial\Omega \setminus K_n\right).$$

We have constructed a sequence polynomials $\{p_k\}_{k\in\mathbb{N}}$ which fulfills the properties (1)-(3).

Since (for all $N \in \mathbb{N}$):

$$1 \geq \sum_{n=1}^{\infty} \sigma\left(K_{n+1} \setminus K_n\right) \geq \sum_{n=1}^{\infty} \frac{1}{2N_0} \sigma\left(\partial\Omega \setminus K_n\right) \geq \sum_{n=1}^{N} \frac{1}{2N_0} \sigma\left(\partial\Omega \setminus K_N\right)$$
$$= \frac{N}{2N_0} \sigma\left(\partial\Omega \setminus K_N\right)$$

there exists $N \in \mathbb{N}$ such that $\sigma\left(\partial\Omega \setminus K_N\right) < \tilde{\varepsilon}$. In particular $\sigma\left(K_N\right) > 1 - \tilde{\varepsilon}$ and we can define $K = K_N$ and $p = \sum_{k=1}^{N} p_k$, which now fulfills all the required properties. ∎

Lemma 3. *Let $\varepsilon, a \in (0,1)$ and $m \in \mathbb{N}$. There exists a natural number N and polynomials $p_1, ..., p_N$ such that:*

- *each nonzero term in the expansion of p_n has a degree greater than m and less than the degree of each term in the expansion of p_{n+1},*
- *$|p_n| < a$ on $\partial\Omega$,*
- *$\left|\sum_{k=1}^{N} p_k\right| < 1$ on $\partial\Omega$,*
- *$\sigma\left(z \in \partial\Omega : \left\|\sum_{k=1}^{N} p_k\right\|_z \geq \frac{1}{2}\right) > 1 - \varepsilon$*

Proof. Let $\gamma > 0$ be the number from Lemma 2. We define a sequence of polynomials $\{p_k\}_{k=1}^{\infty}$ with the following properties:

1. each nonzero term in the expansion of p_k has a degree greater than m and less than the degree of each term in the expansion of p_{k+1},
2. $\left|\sum_{j=1}^{k} p_j\right| < 1$ on $\partial\Omega$,
3. $|p_k| < \min\left\{a, 1 - \left|\sum_{j=1}^{k-1} p_j\right|\right\}$ on $\partial\Omega$, $(k > 1)$,

4. if $k > 1$ then the circular, compact set:

$$T_k := \left\{ z \in \partial\Omega : \|p_k\|_z \geq \gamma \left\| \min\left\{ a, 1 - \left| \sum_{j=1}^{k-1} p_j \right| \right\} \right\|_z \right\}$$

has the property: $\sigma(T_k) > 1 - \varepsilon 2^{-k}$.

Let $p_1 = 0$. The properties (1)-(3) are fulfilled for $k = 1$. Now suppose that we have defined $p_1, ..., p_k$ with the properties (1)-(4). Due to Lemma 2 used for the data:

$$\kappa := \max_j \deg p_j, \tilde{\varepsilon} := \varepsilon 2^{-k-1}, h := \min\left\{ a, 1 - \left| \sum_{j=1}^{k} p_j \right| \right\}$$

there exists a polynomial p_{k+1} with the following properties:

- each nonzero monomial in p_{k+1} has a degree greater than κ,
- $|p_{k+1}| < h$ on $\partial\Omega$,
- $\sigma\left(\{z \in \partial\Omega : \|p_{k+1}\|_z \geq \gamma \|h\|_z\}\right) > 1 - \tilde{\varepsilon} = 1 - \varepsilon 2^{-k-1}$.

Now we observe that the properties (1),(3),(4) are obvious. Since:

$$\left| \sum_{j=1}^{k+1} p_j \right| \leq \left| \sum_{j=1}^{k} p_j \right| + |p_{k+1}| < \left| \sum_{j=1}^{k} p_j \right| + 1 - \left| \sum_{j=1}^{k} p_j \right| = 1,$$

we obtain the property (2), which finishes the construction of the sequence $\{p_k\}$.

Let us consider $\{p_k\}_{k=1}^{\infty}$ and $\{T_k\}_{k=2}^{\infty}$ with properties (1)-(4). We can define a compact, circular set $T := \bigcap_{j=2}^{\infty} T_j$ and calculate:

$$\sigma(\partial\Omega \setminus T) \leq \sum_{j=2}^{\infty} \sigma(\partial\Omega \setminus T_j) < \sum_{j=2}^{\infty} \varepsilon 2^{-j} < \varepsilon.$$

In particular $\sigma(T) > 1 - \varepsilon$. Let us consider a sequence of continuous functions: $g_k : T \ni z \mapsto \left\| \sum_{j=1}^{k} p_j \right\|_z$. Since $g_k < 1$ and $g_k \leq g_{k+1}$ there exists $\lim_{k\to\infty} g_k(z) \leq 1$. In particular $\sum_{j=1}^{\infty} \|p_j\|_z^2 \leq 1$, which implies that $\lim_{k\to\infty} \|p_j\|_z = 0$ for $z \in T$. Since $\|p_k\|_z \geq \gamma \left\| \min\left\{ a, 1 - \left| \sum_{j=1}^{k-1} p_j \right| \right\} \right\|_z$ we have $\lim_{k\to\infty} \left\| 1 - \left| \sum_{j=1}^{k-1} p_j \right| \right\|_z = 0$, which gives us $\sum_{j=1}^{\infty} \|p_j\|_z^2 = 1$ for $z \in T$. Since $\{g_k\}$ is a bounded, increasing sequence of continuous functions with limits equal to 1 for all points $z \in T$ therefore the sequence $\{g_k\}$ is uniformly convergent to 1 on T and hence there exists a natural number N such that $g_N \geq \frac{1}{2}$ on T. In particular

$$T \subset \left\{ z \in \partial\Omega : \left\| \sum_{k=1}^{N} p_k \right\|_z \geq \frac{1}{2} \right\},$$

which finishes the proof:

$$\sigma\left(\left\{ z \in \partial\Omega : \left\| \sum_{k=1}^{N} p_k \right\|_z \geq \frac{1}{2} \right\} \right) \geq \sigma(T) > 1 - \varepsilon.$$

■

Now we are able to prove the main **Theorem**:

Proof. Given $j \in \mathbb{N}$ due to Lemma 3 there exist a natural number N_j and nonzero polynomials $p_{j,1}, ..., p_{j,N_j}$ such that

1. each nonzero term in the expansion of $p_{j,i}$ has a degree less than the degree of each term in the expansion of $p_{j,i+1}$ or $p_{j+1,k}$ for all $1 \le k \le N_{j+1}$,
2. $\left|p_{j,i}\right| < 2^{-j}$ on $\partial\Omega$,
3. $\left|\sum_{i=1}^{N_j} p_{j,i}\right| < 1$ on $\partial\Omega$,
4. if $T_j := \left\{z \in \partial\Omega : \left\|\sum_{i=1}^{N_j} p_{j,i}\right\|_z \ge \frac{1}{2}\right\}$ then $\sigma(T_j) > 1 - 2^{-j}$.

Let us define

$$f = \sum_{j=1}^{\infty} \frac{1}{j^2} \sum_{i=1}^{N_j} p_{j,i}.$$

The property (3) guarantees that we have just defined a holomorphic function which is continuous to the boundary.

Given j, i let $I(j,i)$ denotes all degrees of homogeneous polynomials in homogeneous expansion of $p_{j,i}$:

$$p_{j,i} = \sum_{m \in I(j,i)} p_{j,i,m}$$

where $p_{j,i,m}$ denotes a homogeneous polynomial of a degree m. Using these homogeneous polynomials we can obtain the expansion in Taylor coefficients for slice functions of f:

$$f(\lambda z) = \sum_{j=1}^{\infty} \frac{1}{j^2} \sum_{i=1}^{N_j} \sum_{m \in I(j,i)} p_{j,i,m}(z)\lambda^m.$$

Let $s \in (0,2)$. We can observe $\sum_m \left\|p_{j,i,m}\right\|_z^2 = \left\|p_{j,i}\right\|_z^2$. Since $0 < \frac{s}{2} < 1$ we can use a triangle inequality in the metric space $l^{\frac{s}{2}}$ to achieve:

$$\left\|p_{j,i}\right\|_z^s = \left(\left\|p_{j,i}\right\|_z^2\right)^{s/2} = \left(\sum_m \left\|p_{j,i,m}\right\|_z^2\right)^{s/2} \le \sum_m \left(\left\|p_{j,i,m}\right\|_z^2\right)^{s/2} = \sum_m \left\|p_{j,i,m}\right\|_z^s.$$

The property (2) implies: $\left\|p_{j,i}\right\|_z 2^j < 1$ for $z \in \partial\Omega$. Now we can estimate:

$$\begin{aligned}\sum_{j,i,m} \left|j^{-2} p_{j,i,m}(z)\right|^s &= \sum_{j,i,m} j^{-2s} \left\|p_{j,i,m}\right\|_z^s \ge \sum_{j,i} j^{-2s} \left\|p_{j,i}\right\|_z^s \\ &\ge \sum_{j,i} j^{-2s} \left\|p_{j,i}\right\|_z^s \left(\left\|p_{j,i}\right\|_z 2^j\right)^{2-s} = \sum_{j,i} j^{-2s} \left\|p_{j,i}\right\|_z^2 2^{j(2-s)}\end{aligned}$$

for $z \in \partial\Omega$.

Let $D := \bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} T_j$. Since $\sigma\left(\bigcap_{j=k}^{\infty} T_j\right) \geq 1 - \sum_{j=k}^{\infty} 2^{-j} = 1 - 2^{-k+1}$ we have $\sigma(D) = 1$.

Now we can choose $z \in D$. There exists $k(z) \in \mathbb{N}$ such that $z \in \bigcap_{j=k(z)}^{\infty} T_j$. Using the property (4) we can estimate:

$$\sum_{j,i,m} \left| j^{-2} p_{j,i,m}(z) \right|^s \geq \sum_{j,i} j^{-2s} \left\| p_{j,i} \right\|_z^2 2^{j(2-s)} \geq \sum_{j=k(z)}^{\infty} j^{-2s} 2^{j(2-s)} \sum_{i=1}^{N_j} \left\| p_{j,i} \right\|_z^2$$
$$\geq \frac{1}{4} \sum_{j=k(z)}^{\infty} j^{-2s} 2^{j(2-s)} = \infty,$$

which finishes the proof. ■

References

[1] J. Fournier, An interpolation problem for coefficients of H_∞ function, Proc. Amer. Math. Soc. 42 (1972), 402-408.

[2] P. Kot: A Holomorphic Function with Given Almost All Boundary Values on a Domain with Holomorphic Support Function, Journal of Convex Analysis 14, no. 4, 693-704 (2007).

[3] P. Kot: Homogeneous polynomials on strictly convex domains, Proc. Amer. Math. Soc. 135 (2007) 3895-3903.

[4] P. Kot: Bounded holomorphic functions with given maximal modulus on all circles, Proc. Amer. Math. Soc. 137 (2009) 179–187.

[5] P. Kot: About boundary values in $A(\Omega)$. Trans. Amer. Math. Soc. 363 (2011), no. 8, 4063–4079.

[6] A. Pełczyński, Banach spaces of analytic functions and absolutely summing operators, CBMS Regional Conference Series No. 30, Amer. Math. Soc., Providence, R. I. 1977.

[7] P. Wojtaszczyk, On functions in the ball algebra. Proc. Amer. Math. Soc. 85 (1982), no. 2, 184–186.

[8] J. Ryll and P. Wojtaszczyk, On homogeneous polynomials on a complex ball, Trans. Amer. Math. Soc. 276 (1983), 107-116

[9] P. Wojtaszczyk, On values of homogeneous polynomials in discrete sets of points, Studia Math. 84 (1986), 97-104.

[10] W. Rudin, Function theory in the unit ball of C^n, Springer-Verlag Berlin Heidelberg 2008.

Faculty of Applied Mathematics,
AGH University of Science and Technology,
Mickiewicz Avenue 30, 30-059 Cracow,
Poland
emails : Piotr.Kot@agh.edu.pl, mkaras@mat.agh.edu.pl