A potpourri of algebraic properties of the ring of periodic distributions

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Abstract

The set of periodic distributions, with usual addition and convolution, forms a ring, which is isomorphic, via taking a Fourier series expansion, to the ring $\mathcal{S}'(\mathbb{Z}^d)$ of sequences of at most polynomial growth with termwise operations. In this article, we establish several algebraic properties of these rings.

1 Introduction

Purely algebraic properties for rings naturally considered in Analysis, Algebraic Geometry or Operator Theory, have proven to be of significant motivational importance behind theory-building in these areas. For example, the Noetherian property for polynomial rings over a Noetherian ring is the celebrated Hilbert Basis Theorem, which is a cornerstone result in Algebraic Geometry. As a second example, Serre's 1955 question of whether the ring $k[x_1, \cdots, x_n]$ (k a field) is a projective-free ring spurred the development of algebraic K-theory. As a third example, we mention the corona problem: given data a, b in the Hardy algebra $H^\infty(\mathbb{D})$ of bounded holomorphic functions in the unit disk $\mathbb{D}$ in $\mathbb{C}$, Kakutani's 1941 question of whether the pointwise corona condition $|a(z)| + |b(z)| > \delta$ ($z \in \mathbb{D}$) is sufficient for $H^\infty(\mathbb{D})$ to be equal to the ideal $\langle a, b\rangle$ generated by a, b, led to huge advances in Complex Analysis, Function-Theoretic Operator Theory, and Harmonic Analysis through Carleson's 1962 solution to the problem. Moreover, specific algebraic properties possessed by rings arising in various subdomains in

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Mathematics can lead to further advances in the theory. For example, Kazhdan's Property (T) can be established for the special linear group over the ring $\mathcal{O}(X)$ of holomorphic functions by investigating when the special linear group over $\mathcal{O}(X)$ can be generated by elementary matrices.

The theme of this article is to consider a naturally arising ring in Harmonic Analysis and Distribution Theory, namely the ring of periodic distributions, and check which key algebraic properties are possessed by this ring, and which ones aren't. Via a Fourier series expansion, the ring $\mathcal{D}'_{\mathbf{V}}(\mathbb{R}^d)$ of periodic distributions (with usual addition and convolution) is isomorphic to the ring $\mathcal{S}'(\mathbb{Z}^d)$ of sequences of at most polynomial growth with termwise operations, and we recall this below. We will use this in all of our proofs.

1.1 The ring $\mathcal{D}'_{\mathbf{V}}(\mathbb{R}^d)$ of periodic distributions. The ring $\mathcal{S}'(\mathbb{Z}^d)$ of Fourier coefficients of elements of $\mathcal{D}'_{\mathbf{V}}(\mathbb{R}^d)$

For background on periodic distributions and its Fourier series theory, we refer the reader to the books [6, Chapter 16] and [20, p.527-529].

Consider the space $\mathcal{S}'(\mathbb{Z}^d)$ of all complex valued maps on $\mathbb{Z}^d$ of at most polynomial growth, that is,

$$\mathcal{S}'(\mathbb{Z}^d) := \left\{ \mathbf{a} : \mathbb{Z}^d \to \mathbb{C} \;\middle|\; \begin{array}{l} \exists M > 0\ \exists k \in \mathbb{N} \text{ such that} \\ \forall \mathbf{n} \in \mathbb{Z}^d,\ |\mathbf{a}(\mathbf{n})| \leq M(1 + \lvert\mathbf{n}\rvert)^k \end{array} \right\},$$

where $\lvert\mathbf{n}\rvert := |n_1| + \cdots + |n_d|$ for all $\mathbf{n} = (n_1, \cdots, n_d) \in \mathbb{Z}^d$. Then $\mathcal{S}'(\mathbb{Z}^d)$ is a unital commutative ring with pointwise operations, and the multiplicative unit element given by the constant function $\mathbf{n} \mapsto 1$, for all $\mathbf{n} \in \mathbb{Z}^d$. Moreover, $(\mathcal{S}'(\mathbb{Z}^d), +, \cdot)$ is isomorphic as a ring, to the ring $(\mathcal{D}'_{\mathbf{V}}(\mathbb{R}^d), +, *)$, where $\mathcal{D}'_{\mathbf{V}}(\mathbb{R}^d)$ is the set of all periodic distributions (see the definition below), with the usual pointwise addition of distributions, and multiplication taken as convolution of distributions.

For $\mathbf{v} \in \mathbb{R}^d$, the *translation operator* $\mathbf{S_v} : \mathcal{D}'(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$, is defined by

$$\langle \mathbf{S_v}(T), \varphi \rangle = \langle T, \varphi(\cdot + \mathbf{v}) \rangle \text{ for all } \varphi \in \mathcal{D}(\mathbb{R}^d).$$

A distribution $T \in \mathcal{D}'(\mathbb{R}^d)$ is called *periodic with a period* $\mathbf{v} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ if

$$T = \mathbf{S_v}(T).$$

Let $\mathbf{V} := \{\mathbf{v_1}, \cdots, \mathbf{v_d}\}$ be a linearly independent set of d vectors in $\mathbb{R}^d$. We define $\mathcal{D}'_{\mathbf{V}}(\mathbb{R}^d)$ to be the set of all distributions T that satisfy

$$\mathbf{S}_{\mathbf{v_k}}(T) = T, \quad k = 1, \cdots, d.$$

From [5, §34], T is a tempered distribution, and from the above it follows by taking Fourier transforms that $(1 - e^{2\pi i \mathbf{v_k} \cdot \mathbf{y}})\widehat{T} = 0$, for $k = 1, \cdots, d$. It can be seen that

$$\widehat{T} = \sum_{\mathbf{v} \in V^{-1}\mathbb{Z}^d} \alpha_{\mathbf{v}}(T) \delta_{\mathbf{v}},$$

for some scalars $\alpha_{\mathbf{v}}(T) \in \mathbb{C}$, and where V is the matrix with its rows equal to the transposes of the column vectors $\mathbf{v_1}, \cdots, \mathbf{v_d}$:

$$V := \begin{bmatrix} \mathbf{v_1}^\top \\ \vdots \\ \mathbf{v_d}^\top \end{bmatrix}.$$

Also, in the above, $\delta_{\mathbf{v}}$ denotes the usual Dirac measure with support in $\mathbf{v}$:

$$\langle \delta_{\mathbf{v}}, \varphi \rangle = \varphi(\mathbf{v}), \quad \varphi \in \mathcal{D}(\mathbb{R}^d).$$

Then the Fourier coefficients $\alpha_{\mathbf{v}}(T)$ give rise to an element in $\mathcal{S}'(\mathbb{Z}^d)$, and vice versa, every element in $\mathcal{S}'(\mathbb{Z}^d)$ is the set of Fourier coefficients of some periodic distribution. In this manner, the ring $(\mathcal{D}'_{\mathbf{V}}(\mathbb{R}^d), +, *)$ of periodic distributions on $\mathbb{R}^d$ is isomorphic (as a ring) to $(\mathcal{S}'(\mathbb{Z}^d), +, \cdot)$.

The outline of this article is as follows: in the subsequent sections, we will show that the ring $\mathcal{S}'(\mathbb{Z}^d)$ (and hence also the isomorphic ring $\mathcal{D}'_{\mathbf{V}}(\mathbb{R}^d)$) has the following algebraic properties:

1. $\mathcal{S}'(\mathbb{Z}^d)$ is not Noetherian.
2. $\mathcal{S}'(\mathbb{Z}^d)$ is a Bézout ring.
3. $\mathcal{S}'(\mathbb{Z}^d)$ is coherent.
4. $\mathcal{S}'(\mathbb{Z}^d)$ is a Hermite ring.
5. $\mathcal{S}'(\mathbb{Z}^d)$ is not projective-free.
6. For all $m \in \mathbb{N}$, $SL_m(\mathcal{S}'(\mathbb{Z}^d))$ is generated by elementary matrices, that is, $SL_m(\mathcal{S}'(\mathbb{Z}^d)) = E_m(\mathcal{S}'(\mathbb{Z}^d))$.
7. A generalized "corona-type pointwise condition" on the matricial data (A, b) with entries from $\mathcal{S}'(\mathbb{Z}^d)$ for the solvability of $Ax = b$ with x also having entries from $\mathcal{S}'(\mathbb{Z}^d)$.

In each section, we will first give the background of the algebraic property, by recalling key definitions/characterizations, and then prove the property, possibly with additional commentary.

2 Noetherian property

Recall that a commutative ring is called *Noetherian* if every ascending chain of ideals is stationary, that is, given any chain of ideals in the ring:

$$I_1 \subset I_2 \subset I_3 \subset \cdots,$$

there exists a $K \in \mathbb{N}$ such that $I_K = I_{K+1} = \cdots$.

Proposition 2.1. *$\mathcal{S}'(\mathbb{Z}^d)$ is not Noetherian.*

Proof. For $k \in \mathbb{N}$, set $I_k = \{\mathbf{a} \in \mathcal{S}'(\mathbb{Z}^d) : \mathbf{a}(\mathbf{n}) = 0 \text{ for all } |\mathbf{n}| > k\}$. Then I_k is clearly an ideal in $\mathcal{S}'(\mathbb{Z}^d)$. Also, by considering the sequence

$$\mathbf{e}_k := \left(\mathbb{Z}^d \ni \mathbf{n} \mapsto \left\{ \begin{array}{ll} 1 & \text{if } \mathbf{n} = (k, 0, \cdots, 0), \\ 0 & \text{otherwise,} \end{array} \right\} \right) \in \mathcal{S}'(\mathbb{Z}^d),$$

for $k \in \mathbb{N}$, we see that $\mathbf{e}_k \in I_k \setminus I_{k-1}$. So we have the strict inclusions

$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots,$$

showing the existence of an infinite ascending non-stationary chain of ideals. Hence $\mathcal{S}'(\mathbb{Z}^d)$ is not Noetherian. ∎

Remark 2.2. We remark that in the same manner, one can also show that

$$\ell^\infty(\mathbb{Z}^d) := \left\{ \mathbf{a} : \mathbb{Z}^d \to \mathbb{C} \;\middle|\; \begin{array}{l} \exists M > 0 \quad \text{such that} \\ \forall \mathbf{n} \in \mathbb{Z}^d, \ |\mathbf{a}(\mathbf{n})| \leq M \end{array} \right\},$$

the ring of all bounded sequences with pointwise operations, is not Noetherian either.

3 Bézout ring

A commutative ring is called *Bézout* if every finitely generated ideal is principal.

Theorem 3.1. *Every finitely generated ideal in $\mathcal{S}'(\mathbb{Z}^d)$ is principal, that is, $\mathcal{S}'(\mathbb{Z}^d)$ is Bézout ring.*

Before we give the proof of the above result, we collect some useful observations first. For a complex sequence $\mathbf{a} = (\mathbb{Z}^d \ni \mathbf{n} \mapsto \mathbf{a}(\mathbf{n}))$, let

$$|\mathbf{a}|(\mathbf{n}) := |\mathbf{a}(\mathbf{n})|, \quad \mathbf{n} \in \mathbb{Z}^d.$$

Then we can write $\mathbf{a} = |\mathbf{a}| \cdot \mathbf{u_a}$, where

$$\mathbf{u_a}(\mathbf{n}) = \left\{ \begin{array}{cl} \dfrac{\mathbf{a}(\mathbf{n})}{|\mathbf{a}(\mathbf{n})|} & \text{if } \mathbf{a}(\mathbf{n}) \neq 0, \\ 1 & \text{if } \mathbf{a}(\mathbf{n}) = 0. \end{array} \right.$$

Then $\mathbf{u_a} \in \mathcal{S}'(\mathbb{Z}^d)$. Also, $\mathbf{a} \in \mathcal{S}'(\mathbb{Z}^d)$ if and only if $|\mathbf{a}| \in \mathcal{S}'(\mathbb{Z}^d)$. For a complex sequence $\mathbf{a} = (\mathbb{Z}^d \ni \mathbf{n} \mapsto \mathbf{a}(\mathbf{n}))$, let

$$(\mathbf{a}^*)(\mathbf{n}) = \mathbf{a}(\mathbf{n})^*, \quad \mathbf{n} \in \mathbb{Z}^d,$$

where $\mathbf{a}(\mathbf{n})^*$ on the right hand side denotes the complex conjugate of the complex number $\mathbf{a}(\mathbf{n})$. Then $\mathbf{a} \in \mathcal{S}'(\mathbb{Z}^d)$ if and only if $\mathbf{a}^* \in \mathcal{S}'(\mathbb{Z}^d)$. Also, $\mathbf{u_a}\mathbf{u_{a^*}} = \mathbf{1}$ (the constant sequence, taking value 1 everywhere on $\mathbb{Z}^d$) and $|\mathbf{a}| = \mathbf{a}(\mathbf{u_a})^*$.

Proof. It is enough to show that an ideal $\langle \mathbf{a}, \mathbf{b} \rangle$ generated by $\mathbf{a}, \mathbf{b} \in \mathcal{S}'(\mathbb{Z}^d)$ is principal. We'll show that $\langle \mathbf{a}, \mathbf{b} \rangle = \langle |\mathbf{a}| + |\mathbf{b}| \rangle$.

Since $(\mathbf{u_a})^*, (\mathbf{u_b})^* \in \mathcal{S}'(\mathbb{Z}^d)$, we have $|\mathbf{a}| + |\mathbf{b}| = \mathbf{a}(\mathbf{u_a})^* + \mathbf{b}(\mathbf{u_b})^* \in \langle \mathbf{a}, \mathbf{b} \rangle$. Thus $\langle |\mathbf{a}| + |\mathbf{b}| \rangle \subset \langle \mathbf{a}, \mathbf{b} \rangle$.

Define $\boldsymbol{\alpha}$ by

$$\boldsymbol{\alpha}(\mathbf{n}) = \begin{cases} \dfrac{\mathbf{a}(\mathbf{n})}{|\mathbf{a}(\mathbf{n})| + |\mathbf{b}(\mathbf{n})|} & \text{if } |\mathbf{a}(\mathbf{n})| + |\mathbf{b}(\mathbf{n})| \neq 0, \\ 1 & \text{if } |\mathbf{a}(\mathbf{n})| + |\mathbf{b}(\mathbf{n})| = 0, \end{cases}$$

for all $\mathbf{n} \in \mathbb{Z}^d$. Then $|\boldsymbol{\alpha}(\mathbf{n})| \leq 1$ for all $\mathbf{n}$, and so $\boldsymbol{\alpha} \in \mathcal{S}'(\mathbb{Z}^d)$. Moreover, $\mathbf{a} = \boldsymbol{\alpha} \cdot (|\mathbf{a}| + |\mathbf{b}|)$, and so $\mathbf{a} \in \langle |\mathbf{a}| + |\mathbf{b}| \rangle$. Similarly, $\mathbf{b} \in \langle |\mathbf{a}| + |\mathbf{b}| \rangle$ too. Hence $\langle \mathbf{a}, \mathbf{b} \rangle \subset \langle |\mathbf{a}| + |\mathbf{b}| \rangle$.

Consequently, $\langle \mathbf{a}, \mathbf{b} \rangle = \langle |\mathbf{a}| + |\mathbf{b}| \rangle$. This completes the proof. ∎

4 Coherence

A commutative unital ring R is called *coherent* if every finitely generated ideal I is finitely presentable, that is, there exists an exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow I \longrightarrow 0,$$

where F is a finitely generated free R-module and K is a finitely generated R-module.

We refer the reader to the monograph [8] for background on coherent rings and for the relevance of the property of coherence in homological algebra. All Noetherian rings are coherent, but not all coherent rings are Noetherian. For example, the polynomial ring $\mathbb{C}[x_1, x_2, x_3, \cdots]$ is not Noetherian (because the sequence of ideals $\langle x_1 \rangle \subset \langle x_1, x_2 \rangle \subset \langle x_1, x_2, x_3 \rangle \subset \cdots$ is ascending and not stationary), but $\mathbb{C}[x_1, x_2, x_3, \cdots]$ is coherent [8, Corollary 2.3.4]. Some equivalent characterizations of coherent rings are listed below:

1. [3]; [7, Theorem 2.0A, p.404]: Let R be a unital commutative ring. Let $n \in \mathbb{N} := \{1, 2, 3, \cdots\}$ and $F = (f_1, \cdots, f_n) \in R^n$. A *relation* G on F, written $G \in F^\perp$, is an n-tuple $G = (g_1, \cdots, g_n) \in R^n$ such that $g_1 f_1 + \cdots + g_n f_n = 0$. The ring R is coherent if and only if for each $n \in \mathbb{N}$ and each $F \in R^n$, the R-module $F^\perp$ is finitely generated.

2. [8, Definition, p.41, p.44]: Let R be a commutative unital ring. An R-module M is called a *coherent R-module* if it is finitely generated and every finitely generated R-submodule N of M is finitely presented, that is, there exists an exact sequence
$$F_1 \longrightarrow F_0 \longrightarrow N \longrightarrow 0$$
with F_0, F_1 both finitely generated, free R-modules. Recall that an R-module is a *free R-module* if it is isomorphic to a direct sum of copies of R. A commutative unital ring R is coherent if and only if R is a coherent R-module.

Although it is known that Bézout *domains* are automatically coherent, we can't use this fact and Theorem 3.1, since $\mathcal{S}'(\mathbb{Z}^d)$ is *not* a domain: there exist nontrivial zero divisors in $\mathcal{S}'(\mathbb{Z}^d)$. For $\mathbf{a} \in \mathcal{S}'(\mathbb{Z}^d)$, let $Z(\mathbf{a})$ denote the zero set of $\mathbf{a}$, that is,

$$Z(\mathbf{a}) := \{\mathbf{n} \in \mathbb{Z}^d : \mathbf{a}(\mathbf{n}) = 0\}.$$

Let $\mathbf{0} \in \mathcal{S}'(\mathbb{Z}^d)$ denote the constant map $\mathbb{Z}^d \ni \mathbf{n} \mapsto 0$.

Theorem 4.1. *$\mathcal{S}'(\mathbb{Z}^d)$ is a coherent ring.*

Proof. Let I be a finitely generated ideal in $\mathcal{S}'(\mathbb{Z}^d)$. Then I is principal by Theorem 3.1, and so there exists an $\mathbf{a} \in \mathcal{S}'(\mathbb{Z}^d)$ such that $I = \langle \mathbf{a} \rangle$. Let $K = \langle \mathbf{1}_{Z(\mathbf{a})} \rangle$, where $\mathbf{1}_{Z(\mathbf{a})}$ is the indicator function of the zero set of $\mathbf{a}$, that is, for all $\mathbf{n} \in \mathbb{Z}^d$,

$$(\mathbf{1}_{Z(\mathbf{a})})(\mathbf{n}) := \begin{cases} 0 & \text{if } \mathbf{a}(\mathbf{n}) \neq 0, \\ 1 & \text{if } \mathbf{a}(\mathbf{n}) = 0. \end{cases}$$

Then $\mathbf{1}_{Z(\mathbf{a})} \in \mathcal{S}'(\mathbb{Z}^d)$. Moreover, let $\varphi : \mathcal{S}'(\mathbb{Z}^d) \to I$ be the ring homomorphism given by $\varphi(\mathbf{b}) = \mathbf{ab}$, for $\mathbf{b} \in \mathcal{S}'(\mathbb{Z}^d)$. Finally, let $F := \mathcal{S}'(\mathbb{Z}^d) = \langle \mathbf{1} \rangle$. Then we will check that the following sequence is exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & F & \xrightarrow{\varphi} & I & \longrightarrow & 0. \\ & & \| & & \| & & \| & & \\ & & \langle \mathbf{1}_{Z(\mathbf{a})} \rangle & & \mathcal{S}'(\mathbb{Z}^d) & & \langle \mathbf{a} \rangle & & \end{array}$$

The exactness at K and I is clear. So we only need to show that

$$(\ker \varphi :=)\ \{\mathbf{b} \in \mathcal{S}'(\mathbb{Z}^d) : \mathbf{ab} = \mathbf{0}\} = \langle \mathbf{1}_{Z(\mathbf{a})} \rangle.$$

Since $\mathbf{1}_{Z(\mathbf{a})} \in \ker \varphi$, it is clear that $\langle \mathbf{1}_{Z(\mathbf{a})} \rangle \subset \ker \varphi$. It remains to show the reverse inclusion. Suppose that $\mathbf{b} \in \ker \varphi$. Then $\mathbf{a}(\mathbf{n})\mathbf{b}(\mathbf{n}) = 0$ for all $\mathbf{n} \in \mathbb{Z}^d$. Now if $\mathbf{a}(\mathbf{n}) \neq 0$, then $\mathbf{b}(\mathbf{n}) = 0$. Hence

$$\mathbf{b} = \mathbf{1}_{Z(\mathbf{a})} \cdot \mathbf{b} \in \langle \mathbf{1}_{Z(\mathbf{a})} \rangle.$$

So $\ker \varphi \subset \langle \mathbf{1}_{Z(\mathbf{a})} \rangle$ as well. ■

Remark on the coherence of $\ell^\infty(\mathbb{Z}^d)$: The above proof of Theorem 4.1 carries over, mutatis mutandis, to the ring $\ell^\infty(\mathbb{Z}^d)$. Thus we obtain the result:

Theorem 4.2. *$\ell^\infty(\mathbb{Z}^d)$ is a coherent ring.*

This *also* follows from a classical result of Neville [14], which gives a topological characterization of coherence for the ring $C(X;\mathbb{R})$ of all real-valued continuous functions on X.

Proposition 4.3 (Neville).
$C(X;\mathbb{R})$ is coherent if and only if X is basically disconnected.

A topological space X is called *basically disconnected* if for each $f \in C(X;\mathbb{R})$, the *cozero set of* f, $\operatorname{coz}(f) := \{x \in X : f(x) \neq 0\}$, has an open closure.

We will need the complex-valued version of the above result, which can be obtained from the following observation.

Lemma 4.4. *$C(X;\mathbb{C})$ is coherent if and only if $C(X;\mathbb{R})$ is coherent.*

Here $C(X;\mathbb{C})$ denotes the ring of all complex-valued continuous functions on X. We will use [8, Corollary 2.2.2 and 2.2.3, p.43], quoted below.

Proposition 4.5.
If (1) *R is a commutative unital ring,*
(2) *M, N coherent R-modules, and*
(3) *$\varphi : M \to N$ a homomorphism,*
then $\ker \varphi$ *is a coherent R-module.*

Proposition 4.6. *Every finite direct sum of coherent modules is a coherent module.*

Proof. (of Lemma 4.4):
("If" part). Suppose that $C(X;\mathbb{R})$ is a coherent ring. Let $n \in \mathbb{N}$.
Let $\mathbf{f}_1 = \mathbf{a}_1 + i\mathbf{b}_1, \cdots, \mathbf{f}_n = \mathbf{a}_n + i\mathbf{b}_n \in C(X;\mathbb{C})$, where each $\mathbf{a}_j, \mathbf{b}_j \in C(X;\mathbb{R})$.
Set $R := C(X;\mathbb{R})$, $M := C(X;\mathbb{R})^{(2n)\times 1}$, and $N := C(X;\mathbb{R})^{2\times 1}$.
Suppose that $\varphi : M \to N$ is the module homomorphism given by multiplication by the matrix

$$[\Phi] := \left[\begin{array}{cc|c|cc} \mathbf{a}_1 & -\mathbf{b}_1 & \cdots & \mathbf{a}_n & -\mathbf{b}_n \\ \mathbf{b}_1 & \mathbf{a}_1 & \cdots & \mathbf{b}_n & \mathbf{a}_n \end{array}\right].$$

By Proposition 4.6, M, N are coherent $C(X;\mathbb{R})$-modules, since $C(X;\mathbb{R})$ is a coherent ring. Next, by proposition 4.5, $\ker \varphi$ is a coherent $C(X;\mathbb{R})$-module, and in particular, it is finitely generated, say by

$$\begin{bmatrix} \mathbf{c}_1^{(k)} \\ \mathbf{d}_1^{(k)} \\ \vdots \\ \mathbf{c}_n^{(k)} \\ \mathbf{d}_n^{(k)} \end{bmatrix}, \quad k = 1, \cdots, m.$$

Let $\mathbf{g}_1 = \boldsymbol{\alpha}_1 + i\boldsymbol{\beta}_1, \cdots, \mathbf{g}_n = \boldsymbol{\alpha}_n + i\boldsymbol{\beta}_n$ (where each $\boldsymbol{\alpha}_j, \boldsymbol{\beta}_j \in C(X;\mathbb{R})$) be such that

$$\mathbf{f}_1\mathbf{g}_1 + \cdots + \mathbf{f}_n\mathbf{g}_n = \mathbf{0}.$$

Then

$$[\Phi] \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\beta}_1 \\ \vdots \\ \boldsymbol{\alpha}_n \\ \boldsymbol{\beta}_n \end{bmatrix} = \mathbf{0},$$

and so there exist $\gamma_1, \cdots, \gamma_m$ such that

$$\begin{bmatrix} \alpha_1 \\ \beta_1 \\ \vdots \\ \alpha_n \\ \beta_n \end{bmatrix} = \gamma_1 \begin{bmatrix} \mathbf{c}_1^{(1)} \\ \mathbf{d}_1^{(1)} \\ \vdots \\ \mathbf{c}_n^{(1)} \\ \mathbf{d}_n^{(1)} \end{bmatrix} + \cdots + \gamma_m \begin{bmatrix} \mathbf{c}_1^{(m)} \\ \mathbf{d}_1^{(m)} \\ \vdots \\ \mathbf{c}_n^{(m)} \\ \mathbf{d}_n^{(m)} \end{bmatrix}.$$

But then

$$\begin{bmatrix} \mathbf{g}_1 \\ \vdots \\ \mathbf{g}_n \end{bmatrix} = \gamma^{(1)} \begin{bmatrix} \mathbf{c}_1^{(1)} + i\mathbf{d}_1^{(1)} \\ \vdots \\ \mathbf{c}_n^{(1)} + i\mathbf{d}_n^{(1)} \end{bmatrix} + \cdots + \gamma^{(m)} \begin{bmatrix} \mathbf{c}_1^{(m)} + i\mathbf{d}_1^{(m)} \\ \vdots \\ \mathbf{c}_n^{(m)} + i\mathbf{d}_n^{(m)} \end{bmatrix}.$$

Hence we see that $(\mathbf{f}_1, \cdots, \mathbf{f}_n)^\perp$ is contained in the $C(X;\mathbb{C})$-module generated by

$$\begin{bmatrix} \mathbf{c}_1^{(1)} + i\mathbf{d}_1^{(1)} \\ \vdots \\ \mathbf{c}_n^{(1)} + i\mathbf{d}_n^{(1)} \end{bmatrix}, \cdots, \begin{bmatrix} \mathbf{c}_1^{(m)} + i\mathbf{d}_1^{(m)} \\ \vdots \\ \mathbf{c}_n^{(m)} + i\mathbf{d}_n^{(m)} \end{bmatrix}.$$

It is also clear that each of the above columns belongs to $(\mathbf{f}_1, \cdots, \mathbf{f}_n)^\perp$. Hence $(\mathbf{f}_1, \cdots, \mathbf{f}_n)^\perp$ also contains the $C(X;\mathbb{C})$-module generated by the above columns. Consequently, $C(X;\mathbb{C})$ is a coherent ring.

("Only if" part). Now suppose that $C(X;\mathbb{C})$ is a coherent ring. Let $n \in \mathbb{N}$ and

$$\mathbf{A} := (\mathbf{a}_1, \cdots, \mathbf{a}_n) \in C(X;\mathbb{R})^{1\times n}.$$

Suppose that

$$\begin{bmatrix} \mathbf{c}_1^{(k)} + i\mathbf{d}_1^{(k)} \\ \vdots \\ \mathbf{c}_n^{(k)} + i\mathbf{d}_n^{(k)} \end{bmatrix}, \quad k = 1, \cdots, m,$$

generate the $C(X;\mathbb{C})$-module $\mathbf{A}^\perp$, where each $\mathbf{c}_j^{(k)}, \mathbf{d}_j^{(k)} \in C(X;\mathbb{R})$. Consider a $\mathbf{B} = (\mathbf{b}_1, \cdots, \mathbf{b}_n) \in C(X;\mathbb{R})^{1\times n}$ such that

$$\mathbf{a}_1\mathbf{b}_1 + \cdots + \mathbf{a}_n\mathbf{b}_n = \mathbf{0}.$$

Then there exist $\mathbf{p}^{(k)}, \mathbf{q}^{(k)} \in C(X;\mathbb{R})$, $k = 1, \cdots, m$ such that

$$\begin{bmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_n \end{bmatrix} = (\mathbf{p}^{(1)} + i\mathbf{q}^{(1)}) \begin{bmatrix} \mathbf{c}_1^{(1)} + i\mathbf{d}_1^{(1)} \\ \vdots \\ \mathbf{c}_n^{(1)} + i\mathbf{d}_n^{(1)} \end{bmatrix} + \cdots + (\mathbf{p}^{(m)} + i\mathbf{q}^{(m)}) \begin{bmatrix} \mathbf{c}_1^{(m)} + i\mathbf{d}_1^{(m)} \\ \vdots \\ \mathbf{c}_n^{(m)} + i\mathbf{d}_n^{(m)} \end{bmatrix}.$$

Equating real parts, we obtain in particular that

$$\begin{bmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_n \end{bmatrix} = \mathbf{p}^{(1)} \begin{bmatrix} \mathbf{c}_1^{(1)} \\ \vdots \\ \mathbf{c}_n^{(1)} \end{bmatrix} - \mathbf{q}^{(1)} \begin{bmatrix} \mathbf{d}_1^{(1)} \\ \vdots \\ \mathbf{d}_n^{(1)} \end{bmatrix} + \cdots + \mathbf{p}^{(m)} \begin{bmatrix} \mathbf{c}_1^{(m)} \\ \vdots \\ \mathbf{c}_n^{(m)} \end{bmatrix} - \mathbf{q}^{(m)} \begin{bmatrix} \mathbf{d}_1^{(m)} \\ \vdots \\ \mathbf{d}_n^{(m)} \end{bmatrix}.$$

Thus the $C(X;\mathbb{R})$-module $\mathbf{A}^{\perp}$ is contained in the $C(X;\mathbb{R})$-module generated by the $2m$ vectors

$$\begin{bmatrix} \mathbf{c}_1^{(1)} \\ \vdots \\ \mathbf{c}_n^{(1)} \end{bmatrix}, \begin{bmatrix} \mathbf{d}_1^{(1)} \\ \vdots \\ \mathbf{d}_n^{(1)} \end{bmatrix}, \cdots, \begin{bmatrix} \mathbf{c}_1^{(m)} \\ \vdots \\ \mathbf{c}_n^{(m)} \end{bmatrix}, \begin{bmatrix} \mathbf{d}_1^{(m)} \\ \vdots \\ \mathbf{d}_n^{(m)} \end{bmatrix}.$$

On the other hand each of these vectors also lie in the $C(X;\mathbb{R})$-module $\mathbf{A}^{\perp}$, which can be seen immediately by equating the real and imaginary parts in

$$\mathbf{a}_1(\mathbf{c}_1^{(k)} + i\mathbf{d}_1^{(k)}) + \cdots + \mathbf{a}_n(\mathbf{c}_n^{(k)} + i\mathbf{d}_n^{(k)}) = \mathbf{0}, \quad k = 1, \cdots, m.$$

Hence the $C(X;\mathbb{R})$-module $\mathbf{A}^{\perp}$ is finitely generated. Consequently, $C(X;\mathbb{R})$ is coherent too. ∎

In light of Neville's result, Proposition 4.3, the above gives:

Corollary 4.7. *$C(X;\mathbb{C})$ is coherent if and only if X is basically disconnected.*

If X is a topological space, then let $C_b(X;\mathbb{C})$ denote the algebra of bounded continuous complex valued functions on X, endowed with pointwise operations and the supremum norm:

$$\|\mathbf{f}\|_\infty := \sup_{x \in X} |\mathbf{f}(x)|, \quad \mathbf{f} \in C_b(X;\mathbb{C}).$$

Then $C_b(X;\mathbb{C})$ is a C^*-algebra, and its maximal ideal space is βX, the Stone-Čech compactification of X.

Let $\mathbb{Z}^d$ be endowed with the usual Euclidean topology inherited from $\mathbb{R}^d$. Then the C^*-algebra $\ell^\infty(\mathbb{Z}^d) = C_b(\mathbb{Z}^d;\mathbb{C})$ is isomorphic to $C(\beta\mathbb{Z}^d;\mathbb{C})$. But the Stone-Čech compactification $\beta\mathbb{Z}^d$ of the discrete space $\mathbb{Z}^d$ is extremally disconnected (that is, the closure of every open set in it is open), see for example [15, §6.3, p.450], and in particular, also basically disconnected. Using Corollary 4.7, Theorem 4.2 follows: $\ell^\infty(\mathbb{Z}^d) = C_b(\mathbb{Z}^d;\mathbb{C}) = C(\beta\mathbb{Z}^d;\mathbb{C})$ is a coherent ring. This completes the alternative proof of the coherence of $\ell^\infty(\mathbb{Z}^d)$.

Remark on the coherence of $c(\mathbb{Z}^d)$: Let $c(\mathbb{Z}^d)$ be the subring of $\ell^\infty(\mathbb{Z}^d)$ consisting of all convergent complex sequences, that is,

$$c(\mathbb{Z}^d) = \left\{ \mathbf{a} \in \ell^\infty(\mathbb{Z}^d) \;\middle|\; \begin{array}{l} \exists L \in \mathbb{C} \text{ such that } \forall \epsilon > 0 \; \exists N \in \mathbb{N} \text{ such that} \\ \forall \mathbf{n} \in \mathbb{Z}^d \text{ such that } \lVert\mathbf{n}\rVert > N, \; |\mathbf{a}(\mathbf{n}) - L| < \epsilon \end{array} \right\}.$$

The C^*-algebra $c(\mathbb{Z}^d)$ is isomorphic to $C(\alpha\mathbb{Z}^d;\mathbb{C})$, where $\alpha\mathbb{Z}^d$ denotes the Alexandroff one-point compactification of $\mathbb{Z}^d$ (where $\mathbb{Z}^d$ has the usual Euclidean topology on $\mathbb{Z}^d$ inherited from $\mathbb{R}^d$). So in light of Corollary 4.7, the question of coherence of $c(\mathbb{Z}^d)$ boils down to investigating whether or not $\alpha\mathbb{Z}^d$ is basically disconnected.

Theorem 4.8.
(1) *$\alpha\mathbb{Z}^d$ is not basically disconnected.*
(2) *$c(\mathbb{Z}^d)$ is not a coherent ring.*

Proof.
(1) Firstly, the closed sets F of $\alpha\mathbb{Z}^d$ are of the form

- F is a finite set of integer tuples, or
- $F = S \cup \{\infty\}$, where S is an arbitrary subset of the integer tuples.

From here it follows that the function $\mathbf{f} : \alpha\mathbb{Z}^d \to \mathbb{C}$ given by

$$\mathbf{f}(\mathbf{n}) = \begin{cases} 0 & \text{if } |\mathbf{n}| \text{ is even or } |\mathbf{n}| = \infty, \\ \dfrac{1}{|\mathbf{n}|} & \text{if } |\mathbf{n}| \text{ is odd,} \end{cases}$$

is continuous. Indeed, if K is any closed subset of $\mathbb{C}$ not containing 0, then $\mathbf{f}^{-1}(K)$ cannot contain ∞ and it can only contain finitely many integer tuples, making it closed in $\alpha\mathbb{Z}^d$. On the other hand, if K is a closed subset of $\mathbb{C}$ containing 0, then $\mathbf{f}^{-1}(K)$ contains ∞, making it closed. Hence the inverse images of closed sets under K stay closed. So $\mathbf{f} \in C(\alpha\mathbb{Z}^d; \mathbb{C})$. However, the cozero set of $\mathbf{f}$ is

$$\mathrm{coz}(\mathbf{f}) = \{\mathbf{n} \in \alpha\mathbb{Z}^d : \mathbf{f}(\mathbf{n}) \neq 0\} = \{\mathbf{n} \in \mathbb{Z}^d : |\mathbf{n}| \text{ is odd}\},$$

whose closure is $\{\mathbf{n} \in \mathbb{Z}^d : |\mathbf{n}| \text{ is odd}\} \cup \{\infty\}$, which is clearly not open in $\alpha\mathbb{Z}^d$. Hence $\alpha\mathbb{Z}^d$ is not basically connected.

(2) It follows from Corollary 4.7 that $c(\mathbb{Z}^d)$ is not coherent. ∎

We remark that $c(\mathbb{Z}^d)$ is not Noetherian since it is not even coherent.

5 $\mathcal{S}'(\mathbb{Z}^d)$ is Hermite

A notion related to coherence is that of a Hermite ring; see for example [19, p.1026]. The study of Hermite rings arose naturally in the development of algebraic K-theory associated with Serre's conjecture [11].

In the language of modules, a ring R is *Hermite* if every finitely generated stably free R-module is free.

It is known that a commutative unital Bézout ring having Bass stable rank 1 is Hermite [22]. It was shown in [16] that the Bass stable rank of $\mathcal{S}'(\mathbb{Z}^d)$ is 1. As $\mathcal{S}'(\mathbb{Z}^d)$ is a Bézout ring (Proposition 3.1), we have the following:

Theorem 5.1. *$\mathcal{S}'(\mathbb{Z}^d)$ is a Hermite ring.*

6 $\mathcal{S}'(\mathbb{Z}^d)$ is not a projective free ring

A related stricter notion than that of being Hermite, is the notion of a projective free ring.

A commutative unital ring R is *projective free* if every finitely generated projective R-module is free.

Clearly every projective free ring is Hermite, but the converse may not hold. In fact $\mathcal{S}'(\mathbb{Z}^d)$ is such an example: we will show below that $\mathcal{S}'(\mathbb{Z}^d)$ is *not* projective free. We will do this using the following characterization of projective free rings; see [2].

Proposition 6.1. *Let R be a commutative unital ring. Then R is projective free if and only if for every $n \in \mathbb{N}$ and every $P \in R^{n\times n}$ such that $P^2 = P$, there exists an integer $r \geq 0$, an $S \in R^{n\times n}$, and an $S^{-1} \in R^{n\times n}$ such that $SS^{-1} = I_n$ and*

$$P = S^{-1} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} S.$$

(Here I_r denotes the $r \times r$ identity matrix in $R^{r\times r}$.)

Theorem 6.2. *$\mathcal{S}'(\mathbb{Z}^d)$ is not a projective free ring.*

Proof. Let $R = \mathcal{S}'(\mathbb{Z}^d)$ be projective free. Let $P = \mathbf{p} \in R^{1\times 1}$ be given by

$$\mathbf{p}(\mathbf{n}) = \begin{cases} 1 & \text{if } |\mathbf{n}| \text{ is even,} \\ 0 & \text{if } |\mathbf{n}| \text{ is odd.} \end{cases}$$

Then $P^2 = P$. Since R is projective free, it follows that there are an integer $r \geq 0$, an $S \in R^{1\times 1}$, and an $S^{-1} \in R^{1\times 1}$ such that

$$P = S^{-1} D S,$$

where, since r can only be 0 or 1, we have respectively that $D = \mathbf{0}$ or $\mathbf{1}$. But then $P = \mathbf{0}$ or $P = \mathbf{1}$, and either case is not possible. This contradiction shows that $\mathcal{S}'(\mathbb{Z}^d)$ is not projective free. ∎

7 $SL_m(R) = E_m(R)$ for $R = \mathcal{S}'(\mathbb{Z}^d)$

Let R be a commutative unital ring and $m \in \mathbb{N}$. Then we introduce the following terminology and notation:

(1) I_m denotes the $m \times m$ identity matrix in $R^{m\times m}$, that is the square matrix with all diagonal entries equal to $1 \in R$ and off-diagonal entries equal to $0 \in R$.

(2) $SL_m(R)$ denotes the group of all $m \times m$ matrices M whose entries are elements of R and determinant $\det M = 1$.

(3) An *elementary matrix* $E_{ij}(\alpha)$ over R has the form $E_{ij} = I_n + \alpha \mathbf{e}_{ij}$, where

1. $i \neq j$,
2. $\alpha \in R$, and
3. $\mathbf{e}_{ij}$ is the $m \times m$ matrix whose entry in the ith row and jth column is 1, and all the other entries of $\mathbf{e}_{ij}$ are zeros.

(4) $E_m(R)$ is the subgroup of $SL_m(R)$ generated by the elementary matrices.

A classical question in commutative algebra is the following:

Question 7.1. For all $m \in \mathbb{N}$, is $SL_m(R) = E_m(R)$?

The answer to this question depends on the ring R. For example, if the ring $R = \mathbb{C}$, then the answer is "Yes", and this is an exercise in linear algebra; see for example [1, Exercise 18.(c), page 71]. On the other hand, if R is the polynomial ring $\mathbb{C}[z_1, \cdots, z_d]$ in the indeterminates $z_1, \cdots, z_d$ with complex coefficients, then if $d = 1$, then the answer is "Yes" (this follows from the Euclidean Division Algorithm in $\mathbb{C}[z]$), but if $d = 2$, then the answer is "No", and [4] contains the following example:

$$\begin{bmatrix} 1 + z_1 z_2 & z_1^2 \\ -z_2^2 & 1 - z_1 z_2 \end{bmatrix} \in SL_2(\mathbb{C}[z_1, z_2]) \setminus E_2(\mathbb{C}[z_1, z_2]).$$

For $d \geq 3$, the answer is "Yes", and this is the K_1-analogue of Serre's Conjecture, which is the Suslin Stability Theorem [18]. The case of R being a ring of real/complex valued continuous functions was considered in [21]. For the ring $R = \mathcal{O}(X)$ of holomorphic functions on Stein spaces in $\mathbb{C}^d$, Question 7.1 was posed as an explicit open problem by Gromov in [9], and was solved in [10]. It is known that $SL_m(\ell^\infty(\mathbb{N})) = E_m(\ell^\infty(\mathbb{N}))$; see [12].

We adapt the proof from [12] for answering Question 7.1 for $R = \ell^\infty(\mathbb{N})$, to answer this question for $R = \mathcal{S}'(\mathbb{Z}^d)$. We'll prove below Theorem 7.3, saying that $SL_m(\mathcal{S}'(\mathbb{Z}^d)) = E_m(\mathcal{S}'(\mathbb{Z}^d))$. For a matrix $M = [m_{ij}] \in \mathbb{C}^{m \times m}$, we set

$$\|M\|_\infty := \max_{1 \leq i,j \leq m} |m_{ij}|.$$

Then $\|M_1 M_2\|_\infty \leq m \|M_1\|_\infty \|M_2\|_\infty$ for $M_1, M_2 \in \mathbb{C}^{m \times m}$. Let S_m denote the symmetry group for a set with m elements. For $p \in S_m$, let $\text{sign}(p)$ denote the sign of p.

Lemma 7.2. *There exist maps*

$$\begin{aligned} m &\mapsto \nu(m) : \mathbb{N} \to \mathbb{N}, \\ m &\mapsto C(m) : \mathbb{N} \to (0, \infty), \\ m &\mapsto k(m) : \mathbb{N} \to \mathbb{N}, \end{aligned}$$

such that for every $m \in \mathbb{N}$ and every $A \in SL_m(\mathbb{C})$, there exist elementary matrices $E_1(A), \cdots, E_{\nu(m)}(A)$ such that

$$A = E_1(A) \cdots E_{\nu(m)}(A),$$

and $\|E_n(A)\|_\infty \leq C(m)(1 + \|A\|_\infty)^{k(m)}$ for all $n = 1, \cdots, \nu(m)$.

Proof. First we note that if $A = [a_{ij}]$ is a square matrix with determinant ± 1, then $\|A\|_\infty$ cannot be too small. Indeed, as

$$\pm 1 = \det A = \sum_{p \in S_m} (\operatorname{sign} p) \cdot a_{1p(1)} \cdots a_{mp(m)},$$

we have $\|A\|_\infty \geq \frac{1}{\sqrt[m]{m!}}$.

Now let $A \in SL_m(\mathbb{C})$. Consider first the case that $|a_{11}| = \|A\|_\infty$. So with $a = a_{11}$, we have

$$A = \left[\begin{array}{c|c} a & * \\ \hline * & * \end{array}\right].$$

Now we premultiply the above by

$$E_a = \left[\begin{array}{cc|c} a^{-1} & 0 & \multirow{2}{*}{0} \\ 0 & a & \\ \hline \multicolumn{2}{c|}{0} & I \end{array}\right].$$

As

$$\begin{bmatrix} a^{-1} & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} 1 & 1-a^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1-a \\ 0 & 1 \end{bmatrix},$$

we see that E_a is a product of four elementary matrices. We have now

$$E_a A = \left[\begin{array}{c|c} 1 & * \\ \hline * & * \end{array}\right].$$

Using the entry 1 as a pivot, we can use it to make all other entries in the first row and first column equal to 0. In other words, there exist elementary matrices $E_1^{(\mathrm{r})}, \cdots, E_{m-1}^{(\mathrm{r})}, E_1^{(\mathrm{c})}, \cdots, E_{m-1}^{(\mathrm{c})}$ such that

$$E_{m-1}^{(\mathrm{r})} \cdots E_1^{(\mathrm{r})} E_a A E_1^{(\mathrm{c})} \cdots E_{m-1}^{(\mathrm{c})} = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & A_{m-1} \end{array}\right]. \tag{1}$$

So we have used $m - 1 + 4 + m - 1 = 2(m + 1)$ elementary matrices to obtain this reduction for A. Moreover, we have control on the size of the elementary matrices we have used in terms of the size of A: indeed,

$$\begin{aligned} \|\text{each factor of } E_a\|_\infty &\leq 1 + \max\{|a^{-1}|, |a|\} \\ &\leq 1 + \max\{\|A\|_\infty, \sqrt[m]{m!}\}, \\ \|E_i^{(\mathrm{r})}\|_\infty, \|E_i^{(\mathrm{c})}\|_\infty &\leq \|A\|_\infty m^4 (1 + \max\{\|A\|_\infty, \sqrt[m]{m!}\})^4, \end{aligned}$$

for all $i = 1, \cdots, m-1$. All this we've done assuming $|a_{11}| = \|A\|_\infty$. If this was not the case, then by working in the same manner as above with the entry (i_*, j_*) such that $|a_{i_*, j_*}| = \|A\|_\infty$, we obtain

$$E_{m-1}^{(\mathrm{r})} \cdots E_1^{(\mathrm{r})} E_a A E_1^{(\mathrm{c})} \cdots E_{m-1}^{(\mathrm{c})} = \left[\begin{array}{ccc|c|ccc} & & & 0 & & & \\ & P & & \vdots & & Q & \\ & & & 0 & & & \\ \hline 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \hline & & & 0 & & & \\ & R & & \vdots & & S & \\ & & & 0 & & & \end{array}\right] =: A',$$

where

$$A_{m-1} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}.$$

Clearly $\det A_{m-1} = \pm 1$, and so we can continue this process by using the largest entry of A_{m-1} and using that as a pivot in the matrix A', till we obtain that

$$E_f A E_b = P,$$

where P is a permutation matrix, and E_f is a product of

$$(m-1+4) + (m-2+4) + \cdots + (1+4)$$

elementary matrices, and E_b is a product of $(m-1) + (m-2) + \cdots + 1$ elementary matrices. Also $\det P = (\det E_f)(\det A)(\det E_b) = 1 \cdot 1 \cdot 1 = 1$. But since each of the $m!/2$ even permutation matrices, which belong to $SL_m(\mathbb{C})$ can be expressed as a finite product of elementary matrices with entries that are bounded by constants that depend only on m, we see that our claim is true. ∎

Theorem 7.3. *For all $m \in \mathbb{N}$, $SL_m(\mathcal{S}'(\mathbb{Z}^d)) = E_m(\mathcal{S}'(\mathbb{Z}^d))$.*

Proof. Suppose $\mathbf{A} \in SL_m(\mathcal{S}'(\mathbb{Z}^d))$. For every $\mathbf{n} \in \mathbb{Z}^d$,

$$\mathbf{A}(\mathbf{n}) = E^{[1]}(\mathbf{n}) \cdots E^{[\nu(m)]}(\mathbf{n}), \tag{2}$$

where $E^{[1]}(\mathbf{n}), \cdots, E^{[\nu(m)]}(\mathbf{n})$ are elementary matrices over $\mathbb{C}$, with

$$\|E^{[j]}(\mathbf{n})\|_\infty \leq C(m)(1 + \|\mathbf{A}(\mathbf{n})\|_\infty)^{k(m)}. \tag{3}$$

An elementary matrix $I_m + \alpha \mathbf{e}_{ij}$ is said to be of "type" (i,j). We know that there are $m^2 - m$ different "types" of elementary matrices. We'd like to see $\mathbf{A}$ expressed as a product $\mathbf{E}^{[1]} \cdots \mathbf{E}^{[N]}$ of elements $\mathbf{E}^{[1]}, \cdots, \mathbf{E}^{[N]}$ from $E_m(\mathcal{S}'(\mathbb{Z}^d))$. In light of (2), it seems tempting to define $\mathbf{E}^{[1]}(\mathbf{n}) = E^{[1]}(\mathbf{n})$ etc, but we note that this is not guaranteed to give an element $\mathbf{E}^{[1]}$ in $E_m(\mathcal{S}'(\mathbb{Z}^d))$ because $\mathbf{E}^{[1]}(\mathbf{n}_1) = E^{[1]}(\mathbf{n}_1)$ may not be of the same type as $\mathbf{E}^{[1]}(\mathbf{n}_2) = E^{[1]}(\mathbf{n}_2)$ for distinct $\mathbf{n}_1, \mathbf{n}_2$. To remedy this, the idea now is as follows. We think of the labels of the types of elementary matrices, say $a_1, \cdots, a_{m^2-m}$, as an alphabet, and consider the long word

$$\underbrace{(a_1 \cdots a_{m^2-m})(a_1 \cdots a_{m^2-m}) \cdots (a_1 \cdots a_{m^2-m})}_{\nu(m) \text{ groups}}.$$

And we create a longer, partly redundant, factorization of $\mathbf{A}(\mathbf{n})$ than the one given in (2) using this long word as explained below. Then the *same* sequence of row operations on each $\mathbf{A}(\mathbf{n})$ will produce I_m. So we'll be able to factorize $\mathbf{A}$ into elementary matrices over $\mathcal{S}'(\mathbb{Z}^d)$, "uniformly" instead of "termwise". We now give the technical details below.

We factor

$$\mathbf{A}(\mathbf{n}) = \underbrace{\left(E_1^{[1]}(\mathbf{n}) \cdots E_{m^2-m}^{[1]}(\mathbf{n})\right) \cdots \left(E_1^{[\nu(m)]}(\mathbf{n}) \cdots E_{m^2-m}^{[\nu(m)]}(\mathbf{n})\right)}_{\nu(m) \text{ groups}},$$

where in each of the $\nu(m)$ groupings, all the matrices are identity, except possibly for one: so if we look at the ith grouping, if $E^{[i]}(\mathbf{n})$ is of type α_k, then

$$\begin{aligned} E_k^{[i]}(\mathbf{n}) &= E^{[i]}(\mathbf{n}), \\ E_\ell^{[i]}(\mathbf{n}) &= I_m \text{ for all } \ell \neq k. \end{aligned}$$

(If it happens that $E^{[i]}(\mathbf{n})$ is itself identity, then we put all of the $E_\ell^{[i]}(\mathbf{n}) = I_m$ for all $\ell = 1, \cdots, m^2 - m$.) Now define $\mathbf{E}_j^{[i]} \in E_m(\mathcal{S}'(\mathbb{Z}^d))$ by

$$\mathbf{E}_j^{[i]}(\mathbf{n}) = E_j^{[i]}(\mathbf{n}), \quad i = 1, \cdots, \nu(m), \quad j = 1, \cdots, m^2 - m, \quad \mathbf{n} \in \mathbb{Z}^d.$$

(The fact that we have entries in $\mathcal{S}'(\mathbb{Z}^d)$ follows from the estimate given in (3).) Then

$$\mathbf{A} = \underbrace{\left(\mathbf{E}_1^{[1]} \cdots \mathbf{E}_{m^2-m}^{[1]}\right) \cdots \left(\mathbf{E}_1^{[\nu(m)]} \cdots \mathbf{E}_{m^2-m}^{[\nu(m)]}\right)}_{\nu(m) \text{ groups}}.$$

This completes the proof. ∎

Remark on $SL_m(R) = E_m(R)$ for all $m \in \mathbb{N}$ when $R = c(\mathbb{Z}^d)$:
Using a result given below in Lemma 7.4, which follows from [21, Lemma 9], we will show Theorem 7.5.

Lemma 7.4 ([21]). *Let R be a commutative topological unital ring such that the set of invertible elements of R is open in R. Let $m \in \mathbb{N}$. If $C \in SL_m(R)$ is sufficiently close to I_m, then C belongs to $E_m(R)$.*

Theorem 7.5. *For all $m \in \mathbb{N}$, $SL_m(c(\mathbb{Z}^d)) = E_m(c(\mathbb{Z}^d))$.*

Proof. Let $m \in \mathbb{N}$ and $\mathbf{A} \in SL_m(c(\mathbb{Z}^d))$. Suppose that L_{ij} is the limit of the matrix entry $\mathbf{A}_{ij} \in c(\mathbb{Z}^d)$, and L be the complex $m \times m$ matrix with the entry L_{ij} in ith row and jth column.

Since $\det : \mathbb{C}^{m \times m} \to \mathbb{C}$ is continuous, we have

$$\det L = \det\Big(\lim \mathbf{A}(\mathbf{n})\Big) = \lim \det \mathbf{A}(\mathbf{n}) = \lim 1 = 1.$$

Let $\epsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that for all $\mathbf{n} \in \mathbb{Z}^d$ such that $\lceil \mathbf{n} \rfloor > N$, we have

$$\|\mathbf{A}(\mathbf{n}) - L\|_\infty < \epsilon.$$

Let $\mathbf{B} \in SL_m(c(\mathbb{Z}^d))$ be defined by

$$\mathbf{B}(\mathbf{n}) = \begin{cases} \mathbf{A}(\mathbf{n}) & \text{if } \lceil \mathbf{n} \rfloor \leq N, \\ L & \text{if } \lceil \mathbf{n} \rfloor > N. \end{cases}$$

Since $SL_m(\mathbb{C}) = E_m(\mathbb{C})$, it is clear that L, as well as the finite number of matrices $\mathbf{A}(\mathbf{n})$ with $\lceil \mathbf{n} \rfloor \le N$, can all be written as a product of elementary matrices. Hence it follows that $\mathbf{B} \in E_m(c(\mathbb{Z}^d))$. But

$$\mathbf{B} = \mathbf{A} + \mathbf{B} - \mathbf{A} = \mathbf{A}(\mathbf{I} + \mathbf{A}^{-1}(\mathbf{B} - \mathbf{A})),$$

where $\mathbf{I}(\mathbf{n}) := I_m$ and $\mathbf{A}^{-1}(\mathbf{n}) = (\mathbf{A}(\mathbf{n}))^{-1}$ for all $\mathbf{n} \in \mathbb{Z}^d$. To complete the proof, it suffices to show that $\mathbf{C} := \mathbf{I} + \mathbf{A}^{-1}(\mathbf{B} - \mathbf{A}) \in E_m(c(\mathbb{Z}^d))$. First note that as $\mathbf{A}, \mathbf{B} \in SL_m(c(\mathbb{Z}^d))$, we have $1 = \det \mathbf{A}(\mathbf{n})$ and $1 = \det \mathbf{B}(\mathbf{n})$ for all $\mathbf{n}$. As $\mathbf{AC} = \mathbf{B}$, it follows that also $\det \mathbf{C}(\mathbf{n}) = 1$, and so $\mathbf{C} \in SL_m(c(\mathbb{Z}^d))$. To show $\mathbf{C} \in E_m(c(\mathbb{Z}^d))$, we will use Lemma 7.4 above, with $R = c(\mathbb{Z}^d)$. As $R = c(\mathbb{Z}^d) = C(\mathbb{Z}^d; \mathbb{C})$ is a Banach algebra, the set of invertible elements in R is an open subset of R. We have

$$\mathbf{C} - \mathbf{I} = \mathbf{A}^{-1}(\mathbf{A} - \mathbf{B}),$$

and since $\mathbf{B}$ could have been made as close to $\mathbf{A}$ as we liked ($\|\mathbf{B} - \mathbf{A}\|_\infty < \epsilon$, and $\epsilon > 0$ was arbitrary), it follows that $\mathbf{C}$ can be made as close as we like to $\mathbf{I}$. Hence $\mathbf{C} \in E_m(c(\mathbb{Z}^d))$ by Lemma 7.4. ∎

8 Solvability of $\mathbf{Ax} = \mathbf{b}$

We will show the following:

Theorem 8.1. *Let $\mathbf{A} \in (\mathcal{S}'(\mathbb{Z}^d))^{m \times n}$, $\mathbf{b} \in (\mathcal{S}'(\mathbb{Z}^d))^{m \times 1}$.*
Then the following two statements are equivalent:

1. *There exists an $\mathbf{x} \in (\mathcal{S}'(\mathbb{Z}^d))^{n \times 1}$ such that $\mathbf{Ax} = \mathbf{b}$.*
2. *There exists a $\delta > 0$ and $k > 0$ such that*

$$\forall \mathbf{n} \in \mathbb{Z}^d,\ \forall y \in \mathbb{C}^m,\ \|(\mathbf{A}(\mathbf{n}))^* y\|_2 \ge \delta(1 + \lceil \mathbf{n} \rfloor)^{-k} |\langle y, \mathbf{b}(\mathbf{n}) \rangle_2|.$$

Here $\langle \cdot, \cdot \rangle_2$ denotes the usual Euclidean inner product on $\mathbb{C}^k$, and $\| \cdot \|_2$ is the corresponding induced norm.

Lemma 8.2. *Let*

(1) $A \in \mathbb{C}^{m \times n}$ and $b \in \mathbb{C}^m$,

(2) there exist a $\delta > 0$ such that $\forall y \in \mathbb{C}^m,\ \|A^* y\|_2 \ge \delta |\langle y, b \rangle_2|$.

Then there exists an $x \in \mathbb{C}^n$ such that $Ax = b$ with $\|x\|_2 \le 1/\delta$.

Proof. If $y \in \ker A^*$, then (2) yields $\langle y, b \rangle_2 = 0$. Thus $b \in (\ker A^*)^\perp = \operatorname{ran} A$.

If $y \in \ker AA^*$, then $\|A^* y\|_2^2 = \langle A^* y, A^* y \rangle = \langle AA^* y, y \rangle = \langle 0, y \rangle = 0$. Thus $A^* y = 0$, and so $y \in \ker A^* = (\operatorname{ran} A)^\perp$. Since we had shown above that $b \in \operatorname{ran} A$, we have $\langle b, y \rangle = 0$. But the choice of $y \in \ker AA^*$ was arbitrary, and so $b \in (\ker AA^*)^\perp = \operatorname{ran}(AA^*)^* = \operatorname{ran}(AA^*)$. Hence there exists a $y_0 \in \mathbb{C}^m$ such that $AA^* y_0 = b$. Taking $x := A^* y_0 \in \mathbb{C}^n$, we have $Ax = b$.

If $b = 0$, then we can take $x = 0$, and the estimate on $\|x\|_2$ is obvious. So we assume that $b \neq 0$ and so $A^*y_0 \neq 0$. We have

$$\|A^*y_0\|^2 = \langle A^*y_0, A^*y_0\rangle = \langle y_0, AA^*y_0\rangle = \langle y_0, b\rangle = |\langle y_0, b\rangle| \leq \frac{1}{\delta}\|A^*y_0\|_2.$$

Since $A^*y_0 \neq 0$, we obtain $\|x\|_2 = \|A^*y_0\|_2 \leq 1/\delta$. ■

Proof. (Of Theorem 8.1:)
(1) $\Rightarrow$ (2): As $\mathbf{x} \in (\mathcal{S}'(\mathbb{Z}^d))^{n\times 1}$, there exist $M, k > 0$ such that for all $\mathbf{n} \in \mathbb{Z}^d$, $\|\mathbf{x}(n)\|_2 \leq M(1 + |\mathbf{n}|)^k$. Thus for all $y \in \mathbb{C}^m$ and all $\mathbf{n} \in \mathbb{Z}^d$,

$$\begin{array}{rcl} |\langle y, \mathbf{b}(\mathbf{n})\rangle_2| & = & |\langle y, \mathbf{A}(\mathbf{n})\mathbf{x}(\mathbf{n})\rangle_2| = |\langle (\mathbf{A}(\mathbf{n}))^* y, \mathbf{x}(\mathbf{n})\rangle_2| \\ & \leq & \|(\mathbf{A}(\mathbf{n}))^* y\|_2 \|\mathbf{x}(\mathbf{n})\|_2 \qquad \text{(Cauchy-Schwarz)} \\ & \leq & \|(\mathbf{A}(\mathbf{n}))^* y\|_2 M(1 + |\mathbf{n}|)^k. \end{array}$$

Setting $\delta := 1/M > 0$ and rearranging gives (2).

(2) $\Rightarrow$ (1): Fix $\mathbf{n} \in \mathbb{Z}^d$. Then (2) gives

$$\forall y \in \mathbb{C}^m, \ \|(\mathbf{A}(\mathbf{n}))^* y\|_2 \geq \delta(1 + |\mathbf{n}|)^{-k} |\langle y, \mathbf{b}(\mathbf{n})\rangle_2|.$$

Lemma 8.2 immediately gives an $x \in \mathbb{C}^n$ such that $\mathbf{A}(\mathbf{n})x = \mathbf{b}(\mathbf{n})$, with

$$\|x\|_2 \leq \frac{1}{\delta(1 + |\mathbf{n}|)^{-k}}. \tag{4}$$

Now set $\mathbf{x}(\mathbf{n}) := x$. By changing $\mathbf{n}$ at the outset, we obtain in this manner a map $\mathbf{x} : \mathbb{Z}^d \to \mathbb{C}^n$. Setting $M = 1/\delta > 0$, we have that $\mathbf{x} \in (\mathcal{S}'(\mathbb{Z}^d))^{n\times 1}$ since we obtain from (4) that

$$\forall \mathbf{n} \in \mathbb{Z}^d, \ \|\mathbf{x}(\mathbf{n})\|_2 \leq M(1 + |\mathbf{n}|)^k.$$

Moreover, $\mathbf{A}\mathbf{x} = \mathbf{b}$. This completes the proof. ■

For $\ell^\infty(\mathbb{Z}^d)$, one has the following analogous result, and the same proof goes through, mutatis mutandis:

Theorem 8.3. *Let* $\mathbf{A} \in (\ell^\infty(\mathbb{Z}^d))^{m\times n}$, $\mathbf{b} \in (\ell^\infty(\mathbb{Z}^d))^{m\times 1}$.
Then the following two statements are equivalent:

1. *There exists an* $\mathbf{x} \in (\ell^\infty(\mathbb{Z}^d))^{n\times 1}$ *such that* $\mathbf{A}\mathbf{x} = \mathbf{b}$.

2. *There exists a* $\delta > 0$ *and* $k > 0$ *such that*

$$\forall \mathbf{n} \in \mathbb{Z}^d, \ \forall y \in \mathbb{C}^m, \ \|(\mathbf{A}(\mathbf{n}))^* y\|_2 \geq \delta |\langle y, \mathbf{b}(\mathbf{n})\rangle_2|.$$

We have

$$\ell^\infty(\mathbb{Z}^d) = C_b(\mathbb{Z}^d; \mathbb{C}) = C(\beta\mathbb{Z}^d; \mathbb{C})$$

is a Banach algebra. Moreover, the natural point evaluation complex homomorphisms

$$\ell^\infty(\mathbb{Z}^d) \ni \mathbf{a} \mapsto \mathbf{a}(\mathbf{n}) \in \mathbb{C},$$

constitute a dense set in its maximal ideal space $\beta\mathbb{Z}^d$. Based on this, one may naturally pose the following question:

Question 8.4.
Let R be a commutative, unital, complex Banach algebra.
Suppose that D is a dense set in the maximal ideal space of R with the usual Gelfand topology, and let $\widehat{\cdot}$ denote the Gelfand transform.
Let $\mathbf{A} \in R^{m\times n}$, $\mathbf{b} \in R^{m\times 1}$.
Are the following two statements are equivalent?

1. There exists an $\mathbf{x} \in R^{n\times 1}$ such that $\mathbf{A}\mathbf{x} = \mathbf{b}$.

2. There exists a $\delta > 0$ such that

$$\forall \varphi \in D,\ \forall y \in \mathbb{C}^m,\ \|(\widehat{\mathbf{A}}(\varphi))^* y\|_2 \geq \delta |\langle y, \widehat{\mathbf{b}}(\varphi)\rangle_2|.$$

(Here $\widehat{\mathbf{A}}$, $\widehat{\mathbf{b}}$ denote the matrices comprising the entry-wise Gelfand transforms of $\mathbf{A}$, $\mathbf{b}$ respectively.)

It can be seen easily that (1) $\Rightarrow$ (2) is true. However, we now show that (2) $\Rightarrow$ (1) may not hold, by considering the case of $c(\mathbb{Z}^d) = C(\alpha\mathbb{Z}^d;\mathbb{C})$.

Example 8.5. Let $d = 1$, so that $\mathbb{Z}^d = \mathbb{Z}$, and

$$\mathbf{A}(n) = \begin{bmatrix} 1 & 1 \\ \mathbf{a}_1(n) & \mathbf{a}_2(n) \end{bmatrix} \in \mathbb{R}^{2\times 2}, \quad \mathbf{b}(n) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbb{R}^{2\times 1}, \quad n \in \mathbb{Z},$$

where the (real) sequences $\mathbf{a}_1, \mathbf{a}_2 \in c(\mathbb{Z})$ will be suitably constructed later. Taking the dense set $D = \mathbb{Z}$ in the maximal ideal space $\alpha\mathbb{Z}$ of $c(\mathbb{Z})$, the condition (2) above becomes:

$$\forall n \in \mathbb{Z},\ \forall y = \begin{bmatrix} \eta \\ \zeta \end{bmatrix} \in \mathbb{C}^{2\times 1},\ |\eta + \zeta \mathbf{a}_1(n)|^2 + |\eta + \zeta \mathbf{a}_2(n)|^2 \geq \delta^2 |\eta|^2.$$

If $\eta = 0$, then this condition is trivially satisfied for all $\delta > 0$.
If $\eta \neq 0$, but $\zeta = 0$, then the condition reads $2|\eta|^2 \geq \delta^2|\eta|^2$, which is satisfied for example with $\delta := 1$.
We continue in the rest of this example with $\delta := 1$.
If $\eta \neq 0$ and $\zeta \neq 0$, then dividing throughout by $|\eta|^2$, and by setting $\zeta/\eta = re^{i\theta}$, where $r > 0$ and $\theta \in \mathbb{R}$, we obtain

$$\forall n \in \mathbb{Z},\ \forall r > 0,\ \forall \theta \in \mathbb{R},\ |1 + re^{i\theta}\mathbf{a}_1(n)|^2 + |1 + re^{i\theta}\mathbf{a}_2(n)|^2 \geq \delta^2 = 1,$$

that is,

$$\forall n \in \mathbb{Z},\ \forall r > 0,\ \forall \theta \in \mathbb{R},$$

$$(\mathbf{a}_1(n)^2 + \mathbf{a}_2(n)^2)r^2 + 2(\mathbf{a}_1(n) + \mathbf{a}_2(n))(\cos\theta)r + 1 \geq 0.$$

This will be satisfied for all r, θ, n if, viewed as a (quadratic) polynomial in r (with n, θ fixed arbitrarily), it has no real roots or has coincident real roots, that is, if

$$\Delta := 4\big((\mathbf{a}_1(n) + \mathbf{a}_2(n))^2(\cos\theta)^2 - (\mathbf{a}_1(n)^2 + \mathbf{a}_2(n)^2)\big) \leq 0.$$

First of all, to ensure that we have a quadratic polynomial, we demand that

$$\boxed{\forall n \in \mathbb{Z},\ \mathbf{a}_1(n)^2 + \mathbf{a}_2(n)^2 \neq 0.} \tag{5}$$

Then

$$\begin{aligned}
&\Delta/4 \\
&\quad = (\mathbf{a}_1(n) + \mathbf{a}_2(n))^2 (\cos\theta)^2 - (\mathbf{a}_1(n)^2 + \mathbf{a}_2(n)^2) \\
&\quad = (\mathbf{a}_1(n) + \mathbf{a}_2(n))^2 - (\mathbf{a}_1(n)^2 + \mathbf{a}_2(n)^2) + \big((\cos\theta)^2 - 1\big)(\mathbf{a}_1(n) + \mathbf{a}_2(n))^2 \\
&\quad = 2\mathbf{a}_1(n)\mathbf{a}_2(n) + \big(\underbrace{(\cos\theta)^2 - 1}_{\leq 0}\big)(\mathbf{a}_1(n) + \mathbf{a}_2(n))^2 \leq 2\mathbf{a}_1(n)\mathbf{a}_2(n).
\end{aligned}$$

So we can ensure that $\Delta \leq 0$ by demanding that

$$\boxed{\forall n \in \mathbb{Z},\ \mathbf{a}_1(n) \cdot \mathbf{a}_2(n) \leq 0.} \tag{6}$$

With $\mathbf{a}_1, \mathbf{a}_2$ satisfying (5) and (6), we have that condition (2) holds with $\delta = 1$.

We will now stipulate additional conditions on $\mathbf{a}_1, \mathbf{a}_2$ so that $\mathbf{A}\mathbf{x} = \mathbf{b}$ does *not* possess a solution $\mathbf{x} \in (c(\mathbb{Z}))^{2\times 1}$. To this end, we demand that $\det \mathbf{A}(n) \neq 0$ for all n, that is,

$$\boxed{\forall n \in \mathbb{Z},\ \mathbf{a}_2(n) - \mathbf{a}_1(n) \neq 0.} \tag{7}$$

Then the unique solution $\mathbf{x}(n)$ to $\mathbf{A}(n)\mathbf{x}(n) = \mathbf{b}(n)$ is given by

$$\mathbf{x}(n) = \begin{bmatrix} 1 & 1 \\ \mathbf{a}_1(n) & \mathbf{a}_2(n) \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \dfrac{\mathbf{a}_2(n)}{\mathbf{a}_2(n) - \mathbf{a}_1(n)} \\ \dfrac{\mathbf{a}_1(n)}{\mathbf{a}_1(n) - \mathbf{a}_2(n)} \end{bmatrix}.$$

We want to ensure that $\mathbf{x} := (n \mapsto \mathbf{x}(n))$ does not belong to $(c(\mathbb{Z}))^{2\times 1}$. This will be guaranteed if one of its entries is not a convergent sequence. So we demand, say, that the sequence

$$\boxed{\left(\frac{\mathbf{a}_1(n)}{\mathbf{a}_1(n) - \mathbf{a}_2(n)}\right)_{n\in\mathbb{N}} \text{ does not converge.}} \tag{8}$$

It remains to construct sequences $\mathbf{a}_1, \mathbf{a}_2$ in $c(\mathbb{Z})$ possessing the properties (5), (6), (7), and (8). We may take, for example,

$$\begin{aligned}
\mathbf{a}_1(n) &= \frac{1}{1+n^2},\ n \in \mathbb{Z}, \quad \text{and} \\
\mathbf{a}_2(n) &= -\frac{n \bmod 2}{1+n^2} = \begin{cases} 0 & \text{if } (\mathbb{Z} \ni) n \text{ is even}, \\ -\dfrac{1}{1+n^2} & \text{if } (\mathbb{Z} \ni) n \text{ is odd}. \end{cases}
\end{aligned}$$

Then $\mathbf{a}_1, \mathbf{a}_2 \in c(\mathbb{Z})$ because

$$\lim_{|n|\to\infty} \mathbf{a}_1(n) = 0 = \lim_{|n|\to\infty} \mathbf{a}_2(n).$$

Condition (5) is satisfied since $\forall n \in \mathbb{Z},\ \mathbf{a}_1(n)^2 + \mathbf{a}_2(n)^2 \geq \mathbf{a}_1(n)^2 > 0$.
(6) is fulfilled as

$$\forall n \in \mathbb{Z},\ \mathbf{a}_1(n) \cdot \mathbf{a}_2(n) = -\frac{n \bmod 2}{(1+n^2)^2} \leq 0.$$

Condition (7) holds because $\forall n \in \mathbb{Z},\ \mathbf{a}_1(n) - \mathbf{a}_2(n) = \dfrac{1 + (n \bmod 2)}{1+n^2} > 0$.
Finally, we check that (8) is satisfied too. We have

$$\frac{\mathbf{a}_1(n)}{\mathbf{a}_1(n) - \mathbf{a}_2(n)} = \frac{1}{1 - \mathbf{a}_2(n)/\mathbf{a}_1(n)} = \frac{1}{1 + (n \bmod 2)}.$$

But now we have

$$\begin{aligned} \lim_{n\to\infty} \frac{\mathbf{a}_1(2n)}{\mathbf{a}_1(2n) - \mathbf{a}_2(2n)} &= \frac{1}{1+0} = 1 \\ &\neq \frac{1}{2} = \frac{1}{1+1} = \lim_{n\to\infty} \frac{\mathbf{a}_1(2n+1)}{\mathbf{a}_1(2n+1) - \mathbf{a}_2(2n+1)}, \end{aligned}$$

contradicting the convergence of $\left(\dfrac{\mathbf{a}_1(n)}{\mathbf{a}_1(n) - \mathbf{a}_2(n)} \right)_{n \in \mathbb{N}}$. ■

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