Function Spaces and Nonsymmetric Norm Preserving Maps

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Abstract

Let X, Y be compact Hausdorff spaces and A, B be either closed subspaces of $C(X)$ and $C(Y)$, respectively, containing constants or positive cones of such subspaces. In this paper we study surjections $T : A \longrightarrow B$ satisfying the norm condition $\|T(f)T(g) - 1\|_Y = \|fg - 1\|_X$ for all $f, g \in A$, where $\|\cdot\|_X$ and $\|\cdot\|_Y$ denote the supremum norms. We show that under a mild condition on the strong boundary points of A and B (and the assumption $T(i) = iT(1)$ in the subspace case), the map T is a weighted composition operator on the set of strong boundary points of B. This result is an improvement of the known results for uniform algebra case to closed linear subspaces and their positive cones.

1 Introduction

The problem of characterization of certain preserving maps (linear or not) between some algebras or spaces of functions has been studied extensively. In most cases the preserving condition is related to a norm, the spectrum, the spectral radius or the range of algebra elements. The study of surjections $T : C(X) \longrightarrow C(X)$, not assumed to be linear, which are multiplicatively spectrum-preserving, i.e. $\sigma(T(f)T(g)) = \sigma(fg)$, $f, g \in C(X)$, has been initiated by Molnár in [14] and continued by many authors in many different settings. For instance, multiplicatively spectrum-preserving (or range-preserving)

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maps between uniform algebras and Banach function algebras have been characterized in [6, 8, 9, 15, 16]. Let $\|\cdot\|_X$ denote the supremum norm on $C(X)$. Since in uniform algebras, the Gelfand transformation is an isometry, it follows that any multiplicatively spectrum-preserving map $T : A \longrightarrow B$ between uniform algebras A and B on compact Hausdorff spaces X and Y, satisfies the weaker norm-multiplicative condition $\|T(f)T(g)\|_Y = \|fg\|_X$, for all $f, g \in A$. However, mappings between uniform algebras satisfying this weaker condition (with respect to the supremum norm) have been studied, for instance in [11]. In this case such a map T is a weighted composition operator in modulus on the Choquet boundaries, that is $|T(f)(y)| = |f(\varphi(y))|$ for each f in the domain, where φ is a homeomorphism between the Choquet boundaries.

In [11] Lambert, Luttman, and Tonev also studied surjective maps $T : A \longrightarrow B$ between uniform algebras A and B on compact Hausdorff spaces X and Y, respectively, satisfying $\|T(f)T(g) + 1\|_Y = \|fg + 1\|_X$, $f, g \in A$, called *nonsymmetric multiplicatively norm-preserving* maps. They showed that if T is homogeneous, then it is an isometric algebra isomorphism. Hatori, Miura, and Takagi have characterized maps $T : A \longrightarrow B$ between semisimple commutative Banach algebras that satisfy $r(fg - 1) = r(T(f)T(g) - 1)$ for all $f, g \in A$, where $r(\cdot)$ denotes the spectral radius of f, see [7]. Since in uniform algebra case, this condition is the same as the previous one, the results in [7] provide the description of nonsymmetric norm-preserving surjections between uniform algebras. Indeed, according their results such maps can be described as weighted composition operators on some points of the Choquet boundary and conjugate weighted composition operators on the other points. On the other hand, an alternative description of such a map $T : A \longrightarrow B$ between uniform algebras has been given in [13]. We should note that this description is independent of the points, and characterizes T as an operator. For some related results, see also [2, 10]

In this paper we study nonsymmetric norm-preserving surjections between either closed subspaces of continuous functions containing constants or their positive cones (rather than uniform algebras) and show that under a mild condition they are weighted composition operators on the Choquet boundaries (Theorem 3.1).

2 Preliminaries

Let X be a compact Hausdorff space. By $C(X)$ we mean the Banach algebra of continuous complex-valued functions on X with the supremum norm $\|\cdot\|_X$. For a function $f \in C(X)$, $M(f) = \{x \in X : |f(x)| = \|f\|_X\}$.

A *uniform algebra* on X is a closed subalgebra of $C(X)$ which contains constants and separates the points of X. For a subspace A of $C(X)$, $\mathrm{Ch}(A)$ denotes the Choquet boundary of A which consists of all points $x \in X$ such that the evaluation functional $e_x : A \longrightarrow \mathbb{C}$ defined by $e_x(f) = f(x)$, $f \in A$, is an extreme point of the unit ball of A^*. It is well known that $\mathrm{Ch}(A)$ is a boundary for A, that is for any $f \in A$, there exists $x \in \mathrm{Ch}(A)$ such that $\|f\|_X = |f(x)|$, see [17, Page 184]. For a subset A of $C(X)$, a point $x \in X$ is a *strong boundary point* (*weak peak point*) for A if for each $\epsilon > 0$ and neighborhood V of x there exists $u \in A$ such that

$u(x) = 1 = \|u\|_X$ and $|u| < \epsilon$ on $X \backslash V$. We denote the set of all *strong boundary points* of A by $\Theta(A)$. For a subspace A of $C(X)$ we have $\Theta(A) \subseteq \mathrm{Ch}(A)$ and the equality holds whenever A is a uniform algebra on X, see [12, Theorem 4.7.22].

For each $f \in C(X)$, the peripheral range of f is defined by $R_\pi(f) = \{\lambda \in f(X) : |\lambda| = \|f\|_X\}$. Given a subset A of $C(X)$ we say that $f \in A$ is a peaking function for A if $R_\pi(f) = \{1\}$ and we call a closed subset F of X a peak set of A if there exists a peaking function $f \in A$ with $F = \{x \in X : f(x) = 1\}$. We note that when A is a subspace containing constants or the positive cone of such subspace, replacing u by $\epsilon + (1-\epsilon)u$, we can assume that for $\epsilon > 0$ the above function u associated to a point $x \in \Theta(A)$ and a neighborhood of x, can be chosen to be a peaking function.

It is well known that in a uniform algebra A on X, for any nonempty intersection $\cap_\alpha E_\alpha$ of peak sets of A we have $\cap_\alpha E_\alpha \cap \mathrm{Ch}(A) \neq \emptyset$. However, this also holds true for the case that A is a subspace of $C(X)$, see [4].

For $x \in X$ and $A \subseteq C(X)$, we set

$$V_x(A) = \{f \in A : f(x) = 1 = \|f\|_X\}$$

and

$$F_x(A) = \{f \in A : |f(x)| = 1 = \|f\|_X\}.$$

Clearly these sets are nonempty for all $x \in \Theta(A)$, and they are equal whenever A consists of positive functions.

The following important lemma is well known for uniform algebra case, see for example [18, Lemma 1.1]. However a minor modification of the proof can be applied for closed subspaces containing constants or their positive cones, rather than uniform algebras.

Lemma 2.1. *Let X be a compact Hausdorff space and A be either a closed subspace of $C(X)$ containing the constants or the positive cone of such subspace. Let $x_0 \in \Theta(A)$ and $f \in A$ with $f(x_0) \neq 0$. Then for each $\epsilon > 0$ and any neighborhood U of x_0 there exists a peaking function $u \in V_{x_0}(A)$ such that $R_\pi(fu) = \{f(x_0)\}$, $|fu(x)| < |f(x_0)|$ for each $x \in X$ with $f(x) \neq f(x_0)$ and, furthermore, $|u| < \epsilon$ on $X \backslash U$.*

Proof. We give the sketch of proof, since it is basically the same as [18, Lemma 1.1]. Put

$$F_0 = \{x \in X : |f(x) - f(x_0)| \geq \frac{|f(x_0)|}{2}\}$$

and for each $n \in \mathbb{N}$, put

$$F_n = \{x \in X : \frac{|f(x_0)|}{2^{n+1}} \leq |f(x) - f(x_0)| \leq \frac{|f(x_0)|}{2^n}\}.$$

Clearly for each $n \in \mathbb{N}$, the set $V_n = U \cap (X \backslash (F_0 \cup F_n))$ is an open neighborhood of x_0. Since x_0 is a strong boundary point of A and A contains the constants, there exists a peaking function $u_n \in V_{x_0}(A)$ such that $|u_n| < \min(\frac{|f(x_0)|}{\|f\|_X}, \frac{1}{2^n+1}, \epsilon)$ on $X \backslash V_n$. Then it is easy to see that the function $u = \Sigma_{n=1}^\infty \frac{u_n}{2^n}$ in A has the desired properties. ∎

3 Main Results

In this section we generalize the known results concerning surjective maps $T : A \longrightarrow B$ between uniform algebras A and B on compact Hausdorff spaces X and Y, respectively, satisfying the norm condition

$$\|T(f)T(g) - 1\|_X = \|fg - 1\|_X \quad (f, g \in A)$$

to the case that A and B are closed subspaces of continuous functions or their positive cones.

Our main result is as follows.

Theorem 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be uniform algebras on compact Hausdorff spaces X and Y, respectively. Suppose that $A \subseteq \mathcal{A}$ and $B \subseteq \mathcal{B}$ are either*

(i) *closed subspaces containing constants, or*

(ii) *positive cones of such subspaces.*

Assume, further, that $\Theta(A) = \mathrm{Ch}(\mathcal{A})$ and $\Theta(B) = \mathrm{Ch}(\mathcal{B})$. If $T : A \longrightarrow B$ is a surjective map satisfying $\|T(f)T(g) - 1\|_Y = \|fg - 1\|_X$, for all $f, g \in A$, (and $T(i) = iT(1)$ in case (i)*), then there exists a homeomorphism $\varphi : \mathrm{Ch}(\mathcal{B}) \longrightarrow \mathrm{Ch}(\mathcal{A})$ such that $T(f)(y) = T(1)(y)f(\varphi(y))$ for all $f \in A$ and $y \in \mathrm{Ch}(\mathcal{B})$.*

The proof of this theorem is given through a series of lemmas. Before stating the required lemmas, we prove a theorem concerning surjective multiplicatively norm-preserving maps between subspaces of continuous functions or their positive cones. We should note that the proof of Theorem 3.2 (which is a generalization of similar results for uniform algebra case) is a modification of [4, Theorem 3.5].

Theorem 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be uniform algebras on compact Hausdorff spaces X and Y, respectively. Assume that $A \subseteq \mathcal{A}$ and $B \subseteq \mathcal{B}$ are either*

(i) *subspaces of $C(X)$ and $C(Y)$, respectively, or*

(ii) *positive cones of such subspaces.*

Assume, further, that $\Theta(A) = \mathrm{Ch}(\mathcal{A})$ and $\Theta(B) = \mathrm{Ch}(\mathcal{B})$. If $T : A \longrightarrow B$ is a surjective multiplicatively norm-preserving map, that is $\|T(f)T(g)\|_Y = \|fg\|_X$ for all $f, g \in A$, then there exists a homeomorphism $\varphi : \mathrm{Ch}(\mathcal{B}) \longrightarrow \mathrm{Ch}(\mathcal{A})$ such that $|T(f)(y)| = |f(\varphi(y))|$ for all $f \in A$ and $y \in \mathrm{Ch}(\mathcal{B})$.

Proof. We divide the proof into the following steps.

Step I. For each $y_0 \in \mathrm{Ch}(\mathcal{B})$ and $r > 0$, the intersection $\cap_{T(f) \in rV_{y_0}(B)} M(f) \cap \mathrm{Ch}(\mathcal{A})$ is a singleton independent of r.

Let $y_0 \in \mathrm{Ch}(\mathcal{B})$ and $r > 0$ be given. For each $f_1, ..., f_n \in A$ with $T(f_i) \in rV_{y_0}(B)$ for $i = 1, ..., n$, since T is surjective and in either of cases $\frac{1}{n}\Sigma_{i=1}^n T(f_i) \in B$, there exists $h \in A$ such that $T(h) = \frac{1}{n}\Sigma_{i=1}^n T(f_i)$. Clearly $\|T(f)\|_Y = \|f\|_X$ holds for all $f \in A$. Therefore, $\|h\|_X = \|T(h)\|_Y = r = T(h)(y_0)$. Since $\Theta(A) = \mathrm{Ch}(\mathcal{A})$ and $\mathrm{Ch}(\mathcal{A})$ is a boundary for $\mathcal{A}$, there exists $x_0 \in \Theta(A)$ such that $|h(x_0)| = r = \|h\|_X$. We now show that $x_0 \in M(f_i)$ for $i = 1, ..., n$. Assume on the contrary that $|f_i(x_0)| < r$ for some $1 \leq i \leq n$. Then choosing a neighborhood U of x_0 with

$|f_i| < r$ on U, since $x_0 \in \Theta(A)$ we can find $h' \in V_{x_0}(A)$ such that $|h'| < 1$ on $X \backslash U$. Hence $\|f_i h'\|_X < r$ and consequently $\|T(f_i)\, T(h')\|_Y < r$. Thus for each $y \in Y$

$$\begin{aligned} |T(h')(y)T(h)(y)| &\leq |T(h')(y) \cdot (\frac{1}{n}T(f_i)(y) + \frac{1}{n}\Sigma_{j\neq i}T(f_j)(y))| \\ &\leq \frac{1}{n}|T(f_i)(y) \cdot T(h')(y)| + \frac{(n-1)r}{n} \\ &\leq \frac{1}{n}\|T(f_i)T(h')\|_Y + \frac{(n-1)r}{n} < \frac{r}{n} + \frac{(n-1)r}{n} = r. \end{aligned}$$

Hence $r = |h'(x_0)h(x_0)| \leq \|h'h\|_X = \|T(h')T(h)\|_Y < r$, a contradiction. This argument shows that $x_0 \in \cap_{i=1}^{n} M(f_i)$, that is the family $\{M(f) : T(f) \in rV_{y_0}(B)\}$ of compact subsets of X has finite intersection property and consequently $\cap_{T(f)\in rV_{y_0}(B)} M(f) \neq \emptyset$. Now since $\mathcal{A}$ is a uniform algebra and for each $f \in \mathcal{A}$ and $\lambda \in R_\pi(f)$, the function $f^* = \frac{\|f\|_X^2 + \overline{\lambda} f}{2\|f\|_X^2}$ is a peaking function for $\mathcal{A}$ with $M(f^*) \subseteq M(f)$ we conclude that $\cap_{T(f)\in rV_{y_0}(B)} M(f) \cap \mathrm{Ch}(\mathcal{A}) \neq \emptyset$.

Clearly for each $y_0 \in \mathrm{Ch}(\mathcal{B})$ and $r > 0$ and each point $x_0^r \in \cap_{T(f)\in rV_{y_0}(B)} M(f) \cap \mathrm{Ch}(\mathcal{A})$ we have $T^{-1}(rV_{y_0}(B)) \subseteq rF_{x_0^r}(A)$. We now show that the inclusion $T(rV_{x_0^r}(A)) \subseteq rF_{y_0}(B)$ also holds. We note that, by the same argument as above, $\cap_{f\in rV_{x_0^r}(A)} M(T(f)) \cap \mathrm{Ch}(\mathcal{B}) \neq \emptyset$ and consequently there exists a point $z_0^r \in \mathrm{Ch}(\mathcal{B})$ such that $T(rV_{x_0^r}(A)) \subseteq rF_{z_0^r}(B)$. Hence it suffices to show that $y_0 = z_0^r$. Assume on the contrary that $y_0 \neq z_0^r$. Then considering disjoint neighborhoods U and W of y_0 and z_0^r, respectively, we can find elements $g \in V_{y_0}(B)$ and $h \in V_{z_0^r}(B)$ such that $|g| < 1$ on $Y \backslash U$ and $|h| < 1$ on $Y \backslash W$. Then clearly $\|rg\, h\|_Y < r$. Choosing $f, f' \in A$ with $T(f) = rg$ and $T(f') = h$ we have $f \in T^{-1}(rV_{y_0}(B)) \subseteq rF_{x_0^r}(A)$ and consequently $|f(x_0^r)| = r = \|f\|_X$. In particular, $\alpha f \in rV_{x_0^r}(A)$ for some $\alpha \in \mathbb{T}$ (with $\alpha = 1$ for positive case). Therefore, $T(\alpha f) \in T(rV_{x_0^r}(A)) \subseteq rF_{z_0^r}(B)$, that is $\|T(\alpha f)\|_Y = |(T(\alpha f))(z_0^r)| = r$. This implies that $\|T(\alpha f)h\|_Y = r = |T(\alpha f)(z_0^r)h(z_0^r)|$. Therefore,

$$\|rgh\|_Y = \|T(f)\, T(f')\|_X = \|ff'\|_X = \|\alpha ff'\|_X = \|T(\alpha f)h\|_Y = r$$

which is a contradiction. Hence we have $T^{-1}(rV_{y_0}(B)) \subseteq rF_{x_0^r}(A)$ and $T(rV_{x_0^r}(A)) \subseteq rF_{y_0}(B)$.

Now to complete the proof of this step, it suffices to show that for each $y_0 \in \mathrm{Ch}(\mathcal{B})$ and $r > 0$, if $x_0 \in \cap_{Tf\in V_{y_0}(B)} M(f) \cap \mathrm{Ch}(\mathcal{A})$ and $x_0^r \in \cap_{T(f)\in rV_{y_0}(B)} M(f) \cap \mathrm{Ch}(\mathcal{A})$, then $x_0^r = x_0$. Assume on the contrary that $x_0^r \neq x_0$ for some $r > 0$. As it was noted before, we have $T^{-1}(rV_{y_0}(B)) \subseteq rF_{x_0^r}(A)$ and $T(rV_{x_0^r}(A)) \subseteq rF_{y_0}(B)$ and similarly $T^{-1}(V_{y_0}(B)) \subseteq F_{x_0}(A)$ and $T(V_{x_0}(A)) \subseteq F_{y_0}(B)$. Choosing disjoint neighborhoods of x_0^r and x_0 in X we can find easily functions $f \in V_{x_0}(A)$ and $g \in V_{x_0^r}(A)$ such that $\|rgf\|_X < r$. Since $T(rg) \in rF_{y_0}(B)$ and $T(f) \in F_{y_0}(B)$, we have $r = \|T(rg)T(f)\|_Y = \|(rg)f\|_X < r$, a contradiction.

The above step allows us to define a function $\varphi : \mathrm{Ch}(\mathcal{B}) \longrightarrow \mathrm{Ch}(\mathcal{A})$ which associates to each $y_0 \in \mathrm{Ch}(\mathcal{B})$, the unique point $x_0 \in \cap_{T(f)\in V_{y_0}(B)} M(f) \cap \mathrm{Ch}(\mathcal{A})$.

As we noted before, for each $r > 0$, we have $\cap_{T(f)\in rV_{y_0}(B)} M(f) \cap \mathrm{Ch}(\mathcal{A}) = \{x_0\}$ and consequently $T(rV_{\varphi(y_0)}(A)) \subseteq rF_{y_0}(B)$.

Step II. The equality $|T(f)(y_0)| = |f(\varphi(y_0))|$ holds for all $f \in A$ and $y_0 \in \mathrm{Ch}(\mathcal{B})$.

First assume that $f \in A$ and $y_0 \in \mathrm{Ch}(\mathcal{B})$ such that $|f(\varphi(y_0))| < |Tf(y_0)|$. Since $V = \{x \in X : |f(x)| < |Tf(y_0)|\}$ is an open neighborhood of $\varphi(y_0)$, we can find a function $h \in V_{\varphi(y_0)}(A)$ such that $\|T(f)T(h)\|_Y = \|fh\|_X < |T(f)(y_0)|$. Hence $T(h) \in T(V_{\varphi(y_0)}(A)) \subseteq F_{y_0}(B)$ and

$$|T(f)(y_0)| > \|T(f)T(h)\|_X \geq |T(f)(y_0)||T(h)(y_0)| = |T(f)(y_0)|,$$

which is a contradiction. This argument shows that $|Tf(y_0)| \leq |f(\varphi(y_0))|$ for all $f \in A$ and $y_0 \in \mathrm{Ch}(\mathcal{B})$. The other inequality is proven in a similar manner, that is $|T(f)(y_0)| = |f(\varphi(y_0))|$.

Step III. The function $\varphi : \mathrm{Ch}(\mathcal{B}) \longrightarrow \mathrm{Ch}(\mathcal{A})$ is a homeomorphism.

We first note that φ is a surjective map. Indeed, for each $x_0 \in \mathrm{Ch}(\mathcal{A})$, using the same argument as in Step I, there exists a point $y_0 \in \mathrm{Ch}(\mathcal{B})$ such that

$$\cap_{f\in V_{x_0}(A)} M(T(f)) \cap \mathrm{Ch}(\mathcal{B}) = \{y_0\}.$$

Since, by the definition of φ, $\cap_{T(f)\in V_{y_0}(B)} M(f) \cap \mathrm{Ch}(\mathcal{A}) = \{\varphi(y_0)\}$, the argument given in Step I implies that $\varphi(y_0) = x_0$, i.e. φ is surjective. A similar argument can be applied to show that φ is injective.

To prove the continuity of φ, let $y_0 \in \mathrm{Ch}(\mathcal{B})$ and let U be an open neighborhood of $\varphi(y_0)$ in X. Then there exists $h \in V_{\varphi(y_0)}(A)$ such that $|h| < \frac{1}{2}$ on $X\backslash U$. Using the equality $|T(h)| = |h \circ \varphi|$ on $\mathrm{Ch}(\mathcal{B})$ we conclude that for the open subset $W = \{y \in \mathrm{Ch}(\mathcal{B}) : |T(h)(y)| > \frac{1}{2}\}$ of $\mathrm{Ch}(\mathcal{B})$ we have $\varphi(W) \subseteq U \cap \mathrm{Ch}(\mathcal{A})$. This shows that φ is continuous. Similarly φ^{-1} is also continuous. ■

Now we state the required lemmas for the proof of Theorem 3.1.

In the sequel we assume that X, Y and A, B are as in Theorem 3.1. We also assume that $T : A \longrightarrow B$ is a surjective map satisfying $\|T(f)T(g) - 1\|_Y = \|fg - 1\|_X$, for all $f, g \in A$. Then clearly $T(1)^2 = 1$, that is $T(1) = 1$ whenever B consists of positive functions, and $T(-1), T(1)$ take their values in $\{-1, 1\}$ in the other case. Moreover, in this case $T(\frac{1}{\alpha})T(\alpha) = 1$ for all $\alpha \in \mathbb{C}\backslash\{0\}$, in particular, $T(\alpha)(y) \neq 0$ for all $y \in Y$.

The following identification lemma is well known in uniform algebra case and we state it for our cases.

Lemma 3.3. (i) *Let $f, g \in A$ such that $\|fh - 1\|_X = \|gh - 1\|_X$ holds for all $h \in A$. Then $f = g$.*

(ii) *T is injective and $T(0) = 0$.*

Proof. (i) For each $n \in \mathbb{N}$ it follows from the hypothesis that $\|nfh - 1\|_X = \|ngh - 1\|_X$, that is $\|fh - \frac{1}{n}\|_X = \|gh - \frac{1}{n}\|_X$ for all $h \in A$. Letting $n \to \infty$, we conclude that $\|fh\|_X = \|gh\|_X$ for all $h \in A$. This easily implies that $|f| = |g|$

on $\Theta(A)$. Indeed, if $|f(x_0)| < |g(x_0)|$ for some $x_0 \in \Theta(A)$, then considering an appropriate neighborhood of x_0, since x_0 is a strong boundary point we can find $h \in V_{x_0}(A)$ such that $\|fh\|_X < \|gh\|_X$ which is impossible.

In the case where A and B consist of positive functions, we have $f = g$ on $\Theta(A)$ and consequently $f = g$, since $\Theta(A) = \mathrm{Ch}(\mathcal{A})$ and $\mathrm{Ch}(\mathcal{A})$ is a boundary for $\mathcal{A}$. For the other case, let $x_0 \in \Theta(A)$. If $f(x_0) = 0$, then clearly $f(x_0) = g(x_0) = 0$. So we assume that $f(x_0) \neq 0$. Then for each neighborhood V of x_0, by Lemma 2.1, there exists a peaking function $u \in V_{x_0}(A)$ such that $R_\pi(fu) = \{f(x_0)\}$ and $|u| < 1$ on $X \backslash V$ and, furthermore, $|fu(x)| < |f(x_0)|$ whenever $f(x) \neq f(x_0)$. Clearly $\|\frac{-fu}{f(x_0)} - 1\|_X = 2$ which implies that $\|\frac{-gu}{f(x_0)} - 1\|_X = 2$. Since $|f| = |g|$ on $\Theta(A) = \mathrm{Ch}(\mathcal{A})$ and $\mathrm{Ch}(\mathcal{A})$ is a boundary for the uniform algebra $\mathcal{A}$ we have $\|\frac{g}{f(x_0)}u\|_X = \|\frac{f}{f(x_0)}u\|_X = 1$. This, together with the fact that $\|\frac{-gu}{f(x_0)} - 1\|_X = 2$ implies that $\frac{g(x_1)u(x_1)}{f(x_0)} = 1$ for some $x_1 \in \Theta(A)$ and since $|f(x_1)| = |g(x_1)|$ we conclude that $|f(x_1)u(x_1)| = |f(x_0)|$. Therefore, $f(x_1) = f(x_0)$ which yields $|u(x_1)| = 1$. Hence $x_1 \in V$ and since u is a peaking function we have $u(x_1) = 1$. Thus $g(x_1) = f(x_0)$ and since V is an arbitrary neighborhood of x_0, the continuity of f and g imply that $g(x_0) = f(x_0)$, as desired.

(ii) It is a straightforward consequence of (i). ∎

Lemma 3.4. *Let $r \geq 0$ and $f \in A$. Then $|T(rf)(y)| = r|T(f)(y)|$ holds for all $y \in \Theta(B)$.*

Proof. Let $y \in \Theta(B)$ and assume first that $T(f)(y) = 0$. Then since y is a strong boundary point we can easily find a function $u \in V_y(B)$ such that $\|T(f)u\|_Y < \frac{\epsilon}{r}$. For each $n \in \mathbb{N}$, we put $h_n = nu$. Then clearly $h_n(y) = n$. Choose, by surjectivity of T, $g_n \in A$ such that $T(g_n) = h_n$. Then for each $n \in \mathbb{N}$,

$$\begin{aligned}
n|T(rf)(y)| - 1 &= |h_n(y)T(rf)(y)| - 1 \leq |h_n(y)T(rf)(y) - 1| \\
&\leq \|h_n T(rf) - 1\|_Y = \|g_n r f - 1\|_X \\
&\leq r\|g_n f - 1\|_X + r + 1 = r\|h_n T(f) - 1\|_Y + r + 1 \\
&\leq r\|h_n T(f)\|_Y + 2r + 1 = rn\|uT(f)\|_Y + 2r + 1 \\
&< rn\frac{\epsilon}{r} + 2r + 1 = n\epsilon + 2r + 1.
\end{aligned}$$

Hence $|T(rf)(y)| < \epsilon + \frac{2r+2}{n}$ for all $n \in \mathbb{N}$. Being $\epsilon > 0$ arbitrary, we get $T(rf)(y) = 0 = rT(f)(y)$. Assume now that $T(f)(y) \neq 0$. Then, using Lemma 2.1, there exists $u \in V_y(B)$ such that $R_\pi(T(f)u) = \{T(f)(y)\}$. For each $n \in \mathbb{N}$, as before, we put $h_n = nu$ and choose $g_n \in A$ such that $T(g_n) = h_n$. Then the same argument shows that

$$n|T(rf)(y)| - 1 \leq rn\,\|uT(f)\|_Y + 2r + 1 = rn\,|T(f)(y)| + 2r + 1.$$

Hence $|T(rf)(y)| \leq r|T(f)(y)| + \frac{2r+2}{n}$ for all $n \in \mathbb{N}$. This implies $|T(rf)(y)| \leq r|T(f)(y)|$ for all $r > 0$. Hence

$$|T(f)(y)| = |T(\frac{1}{r}rf)(y)| \leq \frac{1}{r}|T(rf)(y)| \leq |T(f)(y)|,$$

which proves that $|T(rf)(y)| = r|T(f)(y)|$. ∎

Lemma 3.5. *T is multiplicatively norm-preserving, that is $\|T(f)T(g)\|_Y = \|fg\|_X$ for each $f, g \in A$.*

Proof. Let $n \in \mathbb{N}$. Then

$$n\|fg\|_X - 1 \leq \|nfg - 1\|_X = \|T(nf)T(g) - 1\|_Y \leq \|T(nf)T(g)\|_Y + 1.$$

Since $\Theta(B) = \mathrm{Ch}(\mathcal{B})$ and $\mathrm{Ch}(\mathcal{B})$ is a boundary for $\mathcal{B}$, there exists $y_0 \in \Theta(B)$ such that $\|T(nf)T(g)\|_Y = |T(nf)T(g)(y_0)|$. Hence, by Lemma 3.4, $\|T(nf)T(g)\|_Y = n|T(f)\, T(g)(y_0)|$ and consequently

$$n\|fg\|_X - 1 \leq n|T(f)T(g)(y_0)| + 1 \leq n\|T(f)T(g)\|_Y + 1.$$

Thus $\|fg\|_X \leq \|T(f)T(g)\|_Y + \frac{2}{n}$ for all $n \in \mathbb{N}$, that is $\|fg\|_X \leq \|T(f)T(g)\|_Y$. The other inequality also holds, since T^{-1} has the same properties as T. ■

By the above lemma and Theorem 3.2, there exists a homeomorphism φ : $\mathrm{Ch}(\mathcal{B}) \longrightarrow \mathrm{Ch}(\mathcal{A})$ such that $|T(f)(y)| = |f(\varphi(y))|$ for all $f \in A$ and $y \in \mathrm{Ch}(\mathcal{B})$. Clearly, in the case that A and B consist of positive functions, we have $T(f)(y) = f(\varphi(y)) = T(1)(y)f(\varphi(y))$ for all $f \in A$ and $y \in \mathrm{Ch}(\mathcal{B})$. So in the next lemmas we consider the other case that A and B are closed subspaces of continuous functions.

Lemma 3.6. (i) *For $f, g \in A$ we have $-1 \in R_\pi(fg)$ if and only if $-1 \in R_\pi(T(f)\, T(g))$.*
(ii) *For each $\alpha \in \mathbb{C}$, $|T(\alpha)(y)| = |\alpha|$ holds for all $y \in Y$.*
(iii) *For each $\alpha, \beta \in \mathbb{C}$, $\mathrm{Re}(\alpha\beta) \leq \mathrm{Re}(T(\alpha)(y)T(\beta)(y))$.*

Proof. (i) Assume first that $-1 \in R_\pi(fg)$. Then

$$2 = \|fg - 1\|_X = \|T(f)\, T(g) - 1\|_Y \leq \|T(f)\, T(g)\|_Y + 1 = \|fg\|_X + 1 = 2,$$

and consequently

$$\|T(f)\, T(g) - 1\|_Y = \|T(f)\, T(g)\|_Y + 1, \text{that is} - 1 \in \mathrm{R_ß(T(f)\,T(g))}$$

(ii) Clearly the equality holds for $\alpha = 0$. So assume that $\alpha \in \mathbb{C}\backslash\{0\}$. Then

$$|\frac{1}{T(\alpha)(y)}| \leq \|\frac{1}{T(\alpha)}\|_Y = \|T(\frac{1}{\alpha})\|_Y = \frac{1}{|\alpha|}$$

Thus

$$|\alpha| \leq |T(\alpha)(y)| \leq \|T(\alpha)\|_Y = |\alpha|,$$

and the desired equality holds.

(iii) For each $y \in Y$, since

$$0 \leq |T(\alpha)(y)T(\beta)(y) - 1| \leq \|T(\alpha)T(\beta) - 1\|_Y = |\alpha\beta - 1|$$

it follows from (ii) that $\mathrm{Re}(\alpha\beta) \leq \mathrm{Re}(T(\alpha)(y)T(\beta)(y))$. ■

Lemma 3.7. *If $T(1) = 1$, then $T(-1) = -1$.*

Proof. Put $E_1 = \{y \in Y : T(-1)(y) = 1\}$. Then since $T(-1)(Y) \subseteq \{1, -1\}$ we have $Y \backslash E_1 = \{y \in Y : T(-1)(y) = -1\}$. Consider the element $h = \frac{1+T(-1)}{2}$ of B and assume on the contrary that $h \neq 0$. We note that $h = 1$ on E_1 and $h = 0$ on $Y \backslash E_1$ and so $T(-1)h = h$. Choosing $g \in A$ with $T(g) = h$ we have $\|g\|_X = \|h\|_X = 1$. For each $\lambda \in R_\pi(g)$, since $-1 \in R_\pi(-\overline{\lambda} g)$ it follows from the above lemma that $-1 \in R_\pi(T(-\overline{\lambda})h)$. Hence $T(-\overline{\lambda})(y_0) = T(-\overline{\lambda})(y_0)h(y_0) = -1$ for some $y_0 \in E_1$. Therefore, $T1(y_0)T(-\overline{\lambda})(y_0) = T(-\overline{\lambda})(y_0) = -1$ and, using the above lemma, we have

$$-1 = \mathrm{Re}(T(1)(y_0)T(-\overline{\lambda})(y_0)) \geq \mathrm{Re}(-\overline{\lambda}),$$

which conclude that $\mathrm{Re}(\lambda) \geq 1$. Since $|\lambda| = 1$ we get $\lambda = 1$, that is $R_\pi(g) = \{1\}$. Therefore, $R_\pi(-g) = \{-1\}$ and so $-1 \in R_\pi(T(-1)T(g)) = R_\pi(T(-1)h)$. Since $T(-1)h = h$ we conclude that $-1 \in R_\pi(h)$, which is impossible. This proves that $T(-1) = -1$. ∎

Lemma 3.8. *If $T(1) = 1$ and $T(i) = i$ then*
(i) *$T(\alpha) = \alpha$ for all $\alpha \in \mathbb{C}$.*
(ii) *$R_\pi(T(f)) = R_\pi(f)$ for all $f \in A$.*

Proof. (i) We note that, by the above lemma, $T(-1) = -1$. Since $T(i)T(-i) = T(i)T(\frac{1}{i}) = 1$ we get $T(-i) = -i$. Hence, using Lemma 3.6(iii) for $\alpha, \beta \in \{\pm 1, \pm i\}$ we get the desired equality.
(ii) This is immediate from (i) and Lemma 3.6(i). ∎

Lemma 3.9. *If $T(1) = 1$ and $T(i) = i$, then $T(f)(y) = f(\varphi(y))$ for each $f \in A$ and $y \in \Theta(B)$.*

Proof. Let $y_0 \in \mathrm{Ch}(\mathcal{B})$ and $x_0 = \varphi(y_0)$. Clearly the equality $T(f)(y_0) = f(\varphi(y_0))$ holds if $f(x_0) = 0$. So assume that $f(x_0) \neq 0$. Choose an arbitrary neighborhood W of y_0 in Y and let V be a neighborhood of x_0 in X with $V \cap \mathrm{Ch}(\mathcal{A}) = \varphi(W \cap \mathrm{Ch}(\mathcal{B})$. Then, by Lemma 2.1, there exists a peaking function $h \in A$ with $h(x_0) = 1 = \|h\|_X$, $|h| < 1$ on $X \backslash V$, $R_\pi(fh) = \{f(x_0)\}$ and $|fh(x)| < |f(x_0)|$ for all $x \in X$ with $f(x) \neq f(x_0)$. Setting $\alpha = f(x_0)$, since $-1 \in R_\pi(-f\alpha^{-1}h)$ we have

$$\|T(f)T(-\alpha^{-1}h) - 1\|_Y = \| - f\alpha^{-1}h - 1\|_X = 2,$$

and so there exists $y_1 \in \mathrm{Ch}(\mathcal{B})$ with $|T(f)(y_1)T(-\alpha^{-1}h)(y_1) - 1| = 2$. This clearly implies that

$$T(f)(y_1)T(-\alpha^{-1}h)(y_1) = -1,$$

since $\|T(f)T(-\alpha^{-1}h)\| = \| - f\alpha^{-1}h\| = 1$. Therefore,

$$|f(\varphi(y_1))\alpha^{-1}h(\varphi(y_1))| = |T(f)(y_1)T(\alpha^{-1}h)(y_1)| = 1,$$

that is $|f(\varphi(y_1))h(\varphi(y_1))| = |\alpha| = |f(x_0)|$. Thus $f(\varphi(y_1)) = f(x_0)$ and consequently $h(\varphi(y_1)) = 1$ and $\varphi(y_1) \in V$, i.e. $y_1 \in W$. Since $R_\pi(T(-\alpha^{-1}h)) = R_\pi(-\alpha^{-1}h) = \{-\alpha^{-1}\}$ and

$$|T(-\alpha^{-1}h)(y_1)| = |\alpha^{-1}h(\varphi(y_1))| = |\alpha^{-1}| = \|T(-\alpha^{-1}h)\|_Y,$$

we get $T(-\alpha^{-1}h)(y_1) = -\alpha^{-1}$. This implies that $T(f)(y_1) = \alpha = f(x_0) = f(\varphi(y_0))$. Being W arbitrary, it follows from the continuity of $T(f)$ that $T(f)(y_0) = f(\varphi(y_0))$, as desired. ∎

Proof of Theorem 3.1. The proof has been completed for the case that A and B are positive cones. For the other case, since $T(1)(Y) \subseteq \{1, -1\}$ and, by assumption, $T(i) = iT(1)$ it follows that $T(i)(Y) \subseteq \{i, -i\}$. Consider the closed subspace $B_1 = T(1)B$ of $\mathcal{B}$ which contains the constants, since $T(1)^2 = 1$. It is easy to see that $\Theta(B_1) = \Theta(B) = \mathrm{Ch}(\mathcal{B})$. Let $S : A \longrightarrow B_1$ be defined by $S(f) = T(1)\, T(f)$, $f \in A$. Using the facts that $T(i) = iT(1)$ and $T(1)^2 = 1$, it can be easily shown that S is a surjective map satisfying $S(1) = 1$, $S(i) = i$ and $\|S(f)S(g) - 1\|_Y = \|fg - 1\|_X$, for all $f, g \in A$. Hence by the above lemma, there exists a homeomorphism $\varphi : \mathrm{Ch}(\mathcal{B}) \longrightarrow \mathrm{Ch}(\mathcal{A})$ such that $S(f)(y) = f(\varphi(y))$ for all $f \in A$ and $y \in \mathrm{Ch}(\mathcal{B})$. This gives the desired description of T.

For a compact Hausdorff space X by a *regular subspace* of $C(X)$ we mean a subspace A of $C(X)$ such that for each $x \in X$ and neighborhood U of x there exists a function $f \in A$ with $f(x) = 1 = \|f\|_X$ and $f = 0$ on $X \backslash U$. In particular, $\Theta(A) = X = \mathrm{Ch}(C(X))$. Hence the next corollary is immediate.

Corollary 3.10. *Let A and B be closed regular subspaces of $C(X)$ and $C(Y)$ which contain constant functions for compact Hausdorff spaces X and Y, respectively. If $T : A \longrightarrow B$ is a surjective map with $T(i) = iT(1)$ satisfying $\|T(f)T(g) - 1\|_Y = \|fg - 1\|_X$ for all $f, g \in A$, then there exists a homeomorphism $\varphi : Y \longrightarrow X$ such that $T(f)(y) = T(1)(y) f(\varphi(y))$ for all $f \in A$ and $y \in Y$.*

For a compact Hausdorff space X, by [3], the kernel of each continuous measure $\mu \in M(X)$ (that is its atomic part is zero), is a (maximal) regular subspace of $C(X)$.

For another example of closed subspaces satisfying the hypotheses of Theorem 3.1, assume that $A_1, ..., A_n$ are uniform algebras on a compact Hausdorff space X with $\cup_{i=1}^{n} \mathrm{Ch}(A_i) = X$ and put $A = \overline{A_1 f_1 + \cdots + A_n f_n} + \mathbb{C}$ where $f_1, ..., f_n \in C(X)$ are strictly positive. Then since $\mathrm{Ch}(A_i) \subseteq \Theta(A_i f_i) \subseteq \Theta(A)$ it follows that A is a closed subspace of $C(X)$ with $\Theta(A) = X = \mathrm{Ch}(C(X))$. Hence we get the next corollary

Corollary 3.11. *Let X be a compact Hausdorff space, $f_1, ..., f_n \in C(X)$ be strictly positive and $A_1, ..., A_n$ be uniform algebras on X with $\cup_{i=1}^{n} \mathrm{Ch}(A_i) = X$. Let $A = \overline{A_1 f_1 + \cdots + A_n f_n} + \mathbb{C}$ and $T : A \longrightarrow A$ be a surjective map with $T(i) = iT(1)$ satisfying $\|T(f)T(g) - 1\|_Y = \|fg - 1\|_X$ for all $f, g \in A$. Then there exists a homeomorphism $\varphi : X \longrightarrow X$ such that $T(f)(y) = T(1)(y) f(\varphi(y))$ for all $f \in A$ and $y \in X$.*

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